Recall

$$\frac{d}{dy}\ln y = \frac{1}{y}.\quad y>0$$

i.e. $\frac{d}{dx}\ln x = \frac{1}{x}.\quad x>0.$

Application: if $u(x)>0$

then $\frac{d}{dx}\ln u(x) = \frac{1}{u(x)}\cdot u'(x)$

Eg 1 $\frac{d}{dx}\ln(x^2+3) = \frac{2x}{x^2+3}$

Rm If $x<0$

$$\frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}.\quad x\neq 0.$$

Eg2. $a > 0$, $\frac{d}{dx} a^x = ?$

Ans: $a^x = (e^{\ln a})^x = e^{(\ln a)x}$

$$\therefore \frac{d}{dx} a^x = \frac{d}{dx} e^{(\ln a)x}$$
$$= e^{(\ln a)x} \cdot \frac{d}{dx}((\ln a)x)$$
$$= a^x \cdot \ln a \quad \cdots (*)$$

Remark: We showed that

$$\frac{d}{dx} a^x = g(a) \cdot a^x$$

where $g(a) = \lim_{h \to 0} \frac{a^h - 1}{h}$

now we know $g(a) \overset{(*)}{=} \ln a$

Eg3 $\frac{d}{dx} 3^{\sin x}$

$= \frac{d}{dx} e^{(\ln 3)\cdot \sin x}$

$= e^{(\ln 3)\sin x} \cdot \frac{d}{dx}((\ln 3)\sin x)$

$= 3^{\sin x} \cdot \ln 3 \cdot \cos x$

Eg4, $a>0$, $a \neq 1$; $x>0$

$\frac{d}{dx} \log_a x = \frac{d}{dx} \frac{\log_e x}{\log_e a}$

$= \frac{d}{dx} \frac{\ln x}{\ln a} = \frac{1}{x \ln a}$

Similarly, $a>0$, $u(x)>0$

$\frac{d}{dx} \log_a u(x) = \frac{u'(x)}{u(x) \ln a}$

Eg 4. $y = \dfrac{(x^2+1)(x+3)^{\frac{1}{2}}}{x-1}, x>1.\ y' = ?$

$$\ln y = \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$$

$\frac{d}{dx}$: $$\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{(x+3)} - \frac{1}{x-1}$$

$$y' = y\cdot\left(\frac{2x}{x^2+1} + \frac{1}{2}\frac{1}{(x+3)} - \frac{1}{x-1}\right)$$

$$= \frac{(x^2+1)(x+3)^{\frac{1}{2}}}{(x-1)}\cdot\left(\frac{2x}{x^2+1} + \frac{1}{2(x+3)} - \frac{1}{x-1}\right)$$

Eg5. $x>0,\ n\in\mathbb{R}$

$$\frac{d}{dx}x^n = \frac{d}{dx}\left(e^{\ln x}\right)^n$$

$$= \frac{d}{dx}e^{n\ln x} = e^{n\ln x}\frac{d}{dx}(n\ln x)$$

$$= x^n \cdot \frac{n}{x} = nx^{n-1}$$

Eg6: $x>0$, $\frac{d}{dx}x^x = ?$

<u>Ans</u> $= \frac{d}{dx}\left(e^{\ln x}\right)^x = \frac{d}{dx}e^{x\ln x}$

$$= e^{x\ln x}\cdot\frac{d}{dx}(x\ln x)$$

$$= x^x\cdot(\ln x+1)$$

Eg7: $e = \lim\limits_{x \to 0} (1+x)^{\frac{1}{x}}$

$$= \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m$$

pf: $\lim\limits_{x \to 0} (1+x)^{\frac{1}{x}}$

$$= \lim_{x \to 0} e^{\frac{\ln(1+x)}{x}}$$

$$= e^{\left(\lim\limits_{x \to 0} \frac{\ln(1+x)}{x}\right)}$$

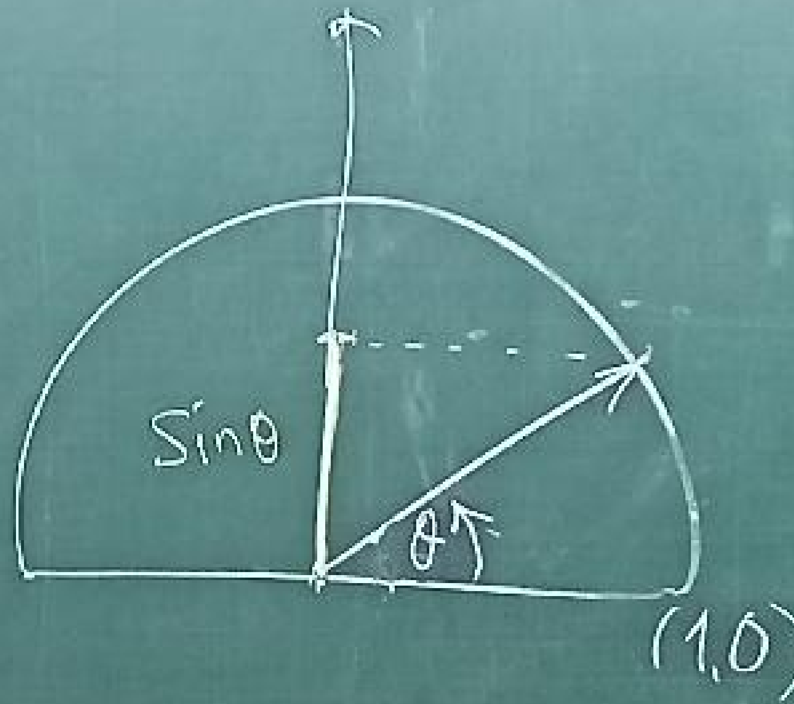
$$= e^{\lim\limits_{x \to 0} \left(\frac{\ln(1+x) - \ln 1}{x - 0}\right)}$$

$$= e^{\frac{d}{dx} \ln(1+x) \Big|_{x=0}} = e^{\frac{1}{1+0}} = e$$

Inverse trigonometric functions

Definition of $\sin^{-1}x$:

$x = \sin\theta$



$$\theta \in \mathbb{R} \xrightarrow{\sin\ (\text{NOT } 1\text{-}1)} x \in [-1,1]$$

Restrict $\downarrow$

$$\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \underset{\sin^{-1}}{\overset{\sin\ (1\text{-}1)}{\rightleftarrows}} x \in [-1,1]$$

$\sin(\sin^{-1}x) = x, \ \forall x \in [-1,1]$

$\sin^{-1}(\sin\theta) = \theta, \ \forall \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Eg1: $\sin^{-1}\left(\frac{1}{2}\right)$

$\sin\theta = \frac{1}{2}, \ \theta = ?$

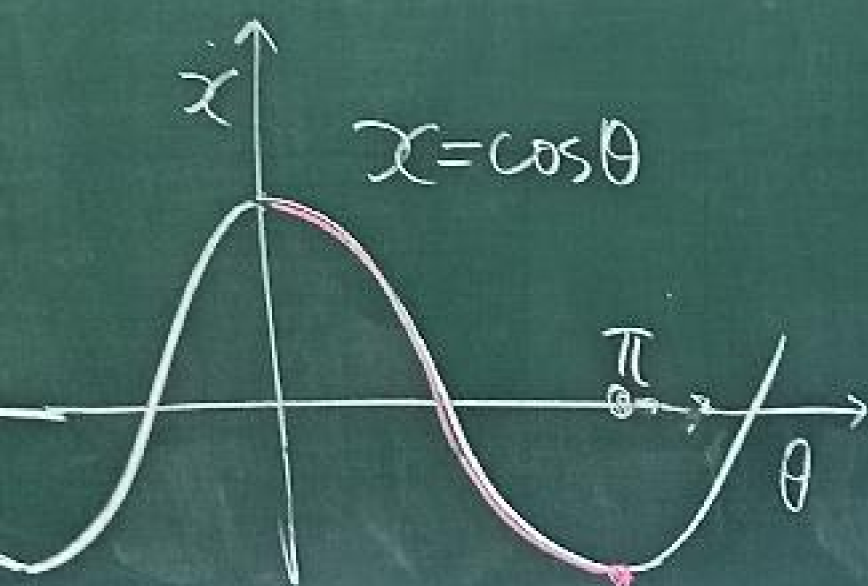
$\theta = \underbrace{\frac{\pi}{6}}_{\sin^{-1}\left(\frac{1}{2}\right)}, \frac{5\pi}{6}, \frac{\pi}{6} \pm 2\pi, \ldots$

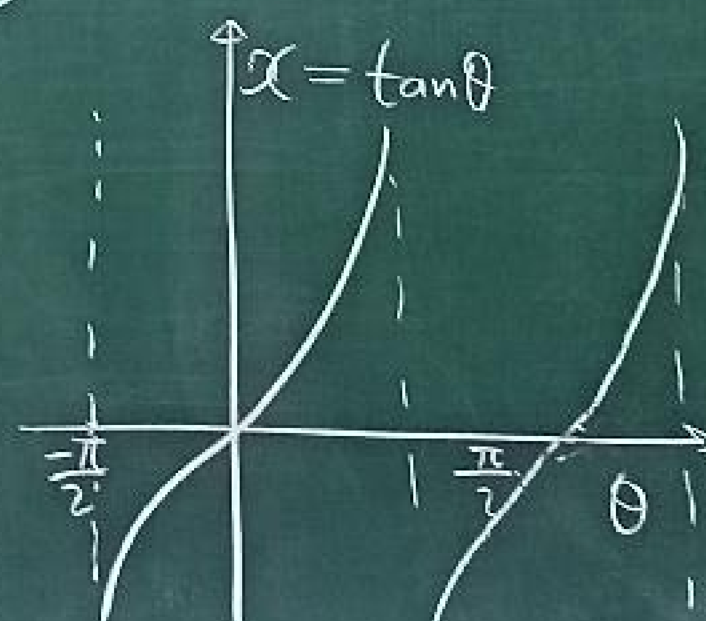
$\because \frac{\pi}{6} \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Eg2: $\underbrace{\sin^{-1}(\sin\pi)}_{=\theta} \neq \pi$

$\begin{cases} \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ \sin\theta = \sin\pi = 0 \end{cases} \Rightarrow \theta = 0$

f	$D_f = R_{f^{-1}}$	$\underset{f^{-1}}{\overset{f}{\rightleftarrows}}$	$R_f = D_{f^{-1}}$	f^{-1}
$\sin$	$\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$		$x \in [-1, 1]$	$\sin^{-1}$
$\cos$	$\theta \in [0, \pi]$		$x \in [-1, 1]$	$\cos^{-1}$
$\tan$	$\theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$		$x \in \mathbb{R}$	$\tan^{-1}$
$\cot$	$\theta \in (0, \pi)$		$x \in \mathbb{R}$	$\cot^{-1}$
$\sec$	$\theta \in [0, \pi] \setminus \left\{\frac{\pi}{2}\right\}$		$x \in (-\infty, -1] \cup [1, \infty)$	$\sec^{-1}$
$\csc$	$\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$		$x \in (-\infty, -1] \cup [1, \infty)$	$\csc^{-1}$





Derivative of Trig. functions

(1) $\frac{d}{dx}\sin^{-1}x = \dfrac{1}{\frac{d}{d\theta}\sin\theta\Big|_{x=\sin\theta}} = \dfrac{1}{\cos\theta}$

$(\theta=\sin^{-1}x)$

It remains to express $\cos\theta$ in terms of x.

$$\begin{cases}\sin\theta = x, \\ \theta\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]\end{cases} \Rightarrow \cos\theta = ?$$

$\sin^2\theta+\cos^2\theta=1$, $\therefore \cos\theta=\pm\sqrt{1-x^2}$

Take "+" $\because \theta\in\left[\frac{-\pi}{2},\frac{\pi}{2}\right]$

$\therefore \frac{d}{dx}\sin^{-1}x = \dfrac{1}{\sqrt{1-x^2}}$, $x\in(-1,1)$

(2) Similarly

$\frac{d}{dx}\cos^{-1}x = \dfrac{1}{-\sin\theta}$ $(\theta\in(0,\pi))$ $= \dfrac{-1}{\sqrt{1-x^2}}$ $x\in(-1,1)$

(3) $\frac{d}{dx}\tan^{-1}x = \frac{1}{\frac{d}{d\theta}\tan\theta} = \frac{1}{\sec^2\theta}$

$(x = \tan\theta)$

$= \frac{1}{1+x^2}, \quad x \in \mathbb{R}$

(4) $\frac{d}{dx}\cot^{-1}x$

: exercise

(5) $\frac{d}{dx}\sec^{-1}x$

(6) $\frac{d}{dx}\csc^{-1}x = \frac{1}{\frac{d}{d\theta}\csc\theta} = \frac{-1}{\csc\theta\cot\theta}$

$(\csc\theta = x)$

$\cot\theta = ?(x)$

$\cot^2\theta = \csc^2\theta - 1 = x^2 - 1$

$\cot\theta = \pm\sqrt{x^2-1}, \quad (|x| > 1)$

"+" or "−" ?

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \begin{matrix} >0 \\ <0 \end{matrix} \text{ if } \theta \in \frac{\mathrm{I}}{\mathrm{IV}}$$

$$\theta \in \frac{\mathrm{I}}{\mathrm{IV}} \iff \begin{matrix} 0 < \theta < \frac{\pi}{2} \\ -\frac{\pi}{2} < \theta < 0 \end{matrix}$$

$$x = \csc\theta$$

$$\iff \begin{matrix} x > 1 \\ x < -1 \end{matrix}$$

$$\therefore \cot\theta = \begin{cases} \sqrt{x^2-1}, & \text{if } x > 1 \\ -\sqrt{x^2-1}, & \text{if } x < -1 \end{cases}$$

$$\because \frac{d}{dx}\csc^{-1}\theta = \frac{-1}{x\left(\pm\sqrt{x^2-1}\right)}$$

$$= \frac{-1}{\begin{cases} +x\sqrt{x^2-1}, & (x>1) \\ -x\sqrt{x^2-1}, & (x<-1) \end{cases}} = \frac{-1}{|x|\sqrt{x^2-1}}$$

TABLE 3.1 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1 + u^2} \frac{du}{dx}$$

4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1 + u^2} \frac{du}{dx}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2 - 1}} \frac{du}{dx}, \quad |u| > 1$$