

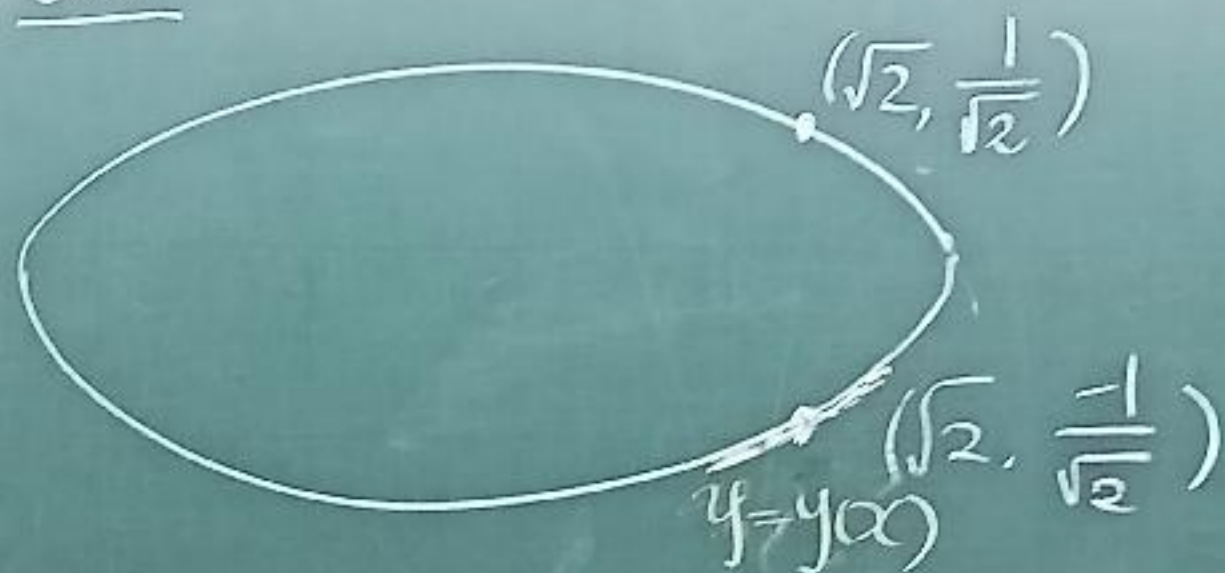
Implicit Differentiation

Goal: Find $\frac{dy}{dx}$ if the function $y(x)$ is not given explicitly, but implicitly by $F(x, y) = 0$.

Ex1: Find the tangent and normal lines of

$$\frac{x^2}{4} + y^2 = 1 \quad \text{at} \quad \left(\sqrt{2}, \frac{-1}{\sqrt{2}}\right)$$

Sol



Method (a) (explicit diff.)

$$y^2 = 1 - \frac{x^2}{4}$$

$$\Rightarrow y(x) = \pm \sqrt{1 - \frac{x^2}{4}}$$

(take "-")

$$\frac{d}{dx} \left(-\sqrt{1 - \frac{x^2}{4}} \right) \Big|_{x=\sqrt{2}} \quad \left(\begin{array}{l} \text{slope of} \\ \text{tangent} \end{array} \right)$$

$$= -\frac{1}{2} \left(1 - \frac{x^2}{4} \right)^{-\frac{1}{2}} \left(-\frac{x}{2} \right) = \frac{1}{2}$$

$\therefore$ tangent: $\frac{y - \frac{-1}{\sqrt{2}}}{x - \sqrt{2}} = \frac{1}{2}$

normal: $\frac{y - \frac{-1}{\sqrt{2}}}{x - \sqrt{2}} = \frac{-1}{(\frac{1}{2})}$

Method (b): (Implicit diff.)

" $\frac{x^2}{4} + y^2 = 1$ implicitly defines $y(x)$ " means

$$\frac{x^2}{4} + y^2(x) \equiv 1$$

for all x near $\sqrt{2}$

$$\xRightarrow{\frac{d}{dx}} \frac{d}{dx} \left(\frac{x^2}{4} + y^2(x) \right) = \frac{d}{dx} 1 = 0$$

$$\Rightarrow \frac{x}{2} + 2y \cdot y' = 0 \quad (*)$$

$$y' \Big|_{(\sqrt{2}, \frac{1}{\sqrt{2}})} = \frac{-x}{4y} \Big|_{(\sqrt{2}, \frac{1}{\sqrt{2}})} = \frac{1}{2}$$

= slope of tangent

$\Rightarrow$ { tangent line
normal line.

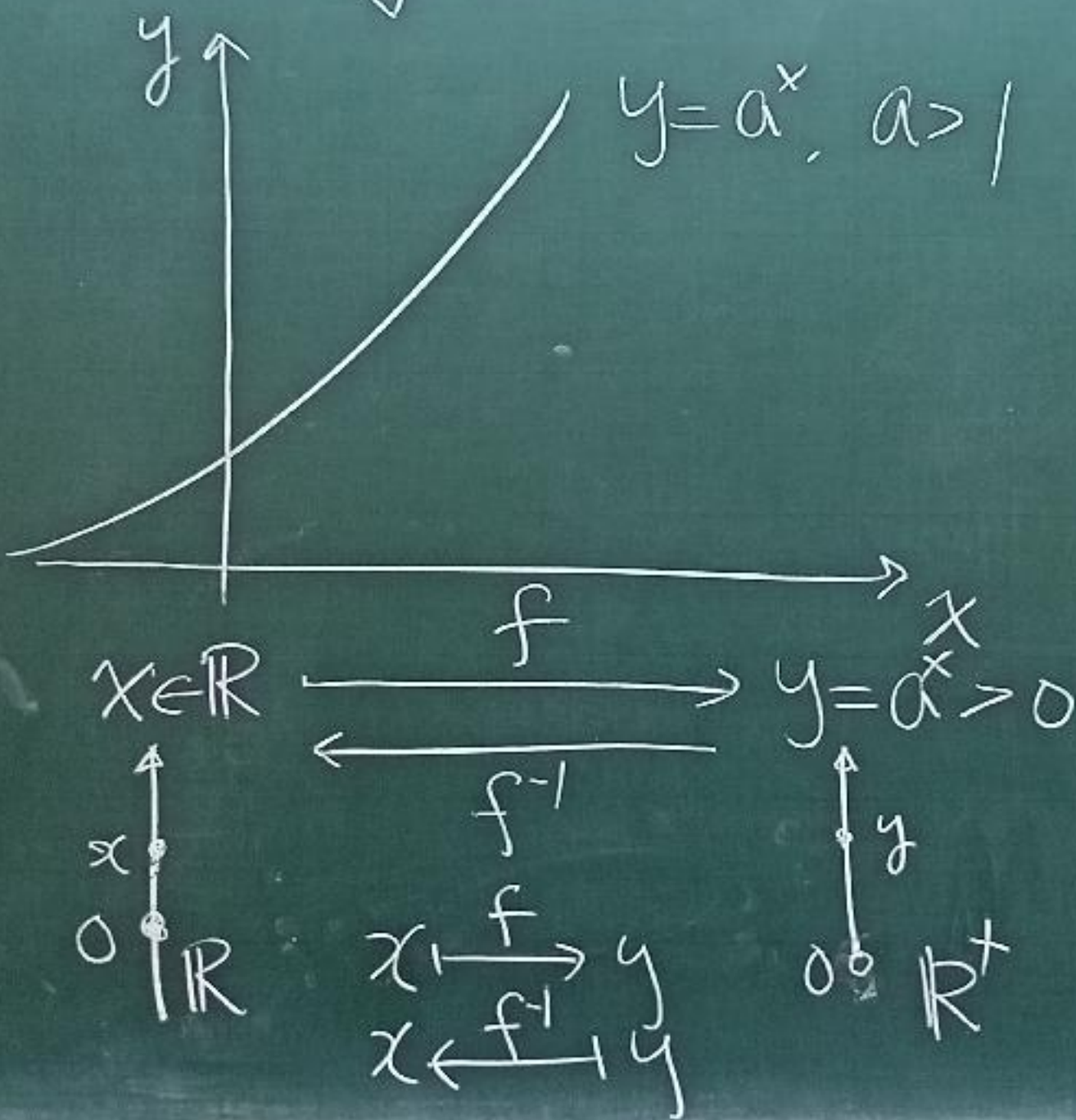
Remark: $y'' \Big|_{(\sqrt{2}, \frac{1}{\sqrt{2}})} = ?$

$$\frac{d}{dx} (*) \Rightarrow \frac{1}{2} + 2(y')^2 + 2y \cdot y'' = 0$$

Substitute $x = \sqrt{2}$, $y = \frac{1}{\sqrt{2}}$

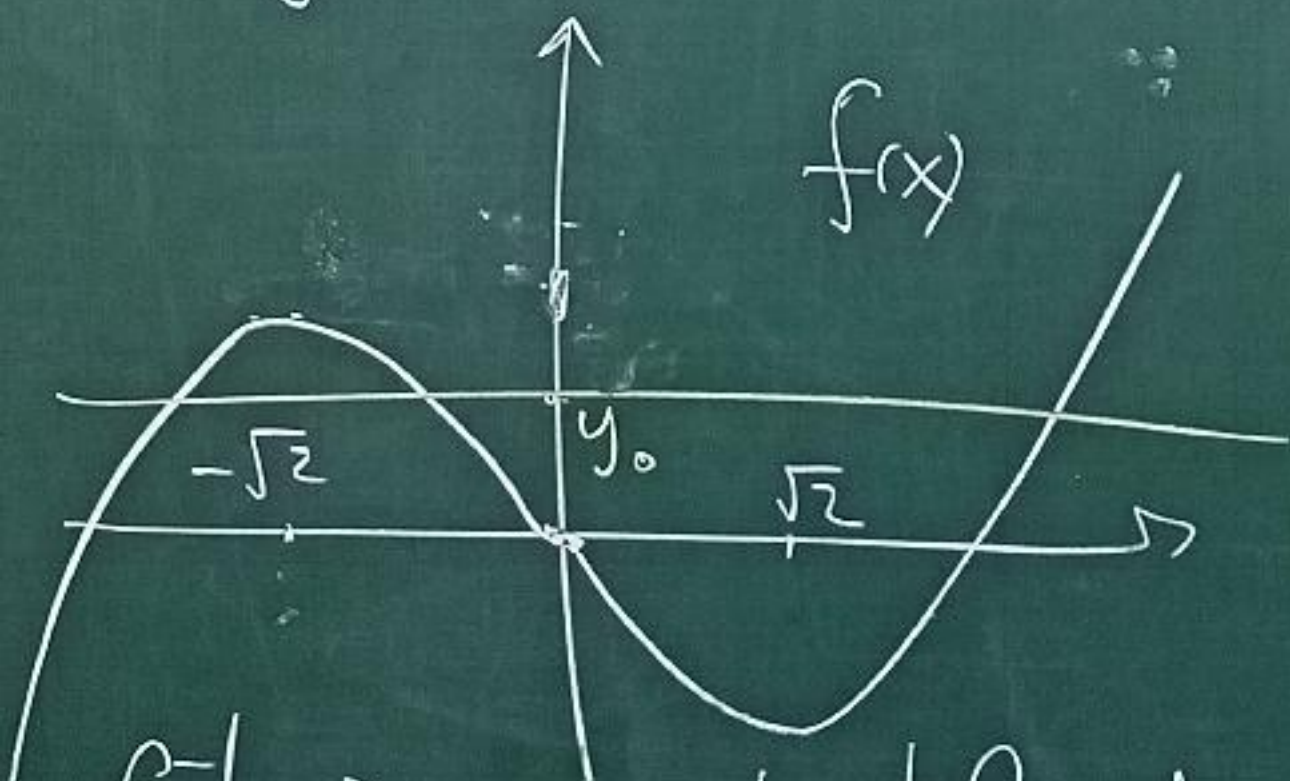
$$y' = \frac{1}{2} \Rightarrow y'' \Big|_{(\sqrt{2}, \frac{1}{\sqrt{2}})} = \frac{1}{\sqrt{2}}$$

Derivative of inverse functions and logarithm.



Remark: If $x_1 \neq x_2$
but $f(x_1) = f(x_2) = y$,
then we can NOT
define $f^{-1}(y)$

Ex 1: $f(x) = x^3 - 3x^2 - 1$

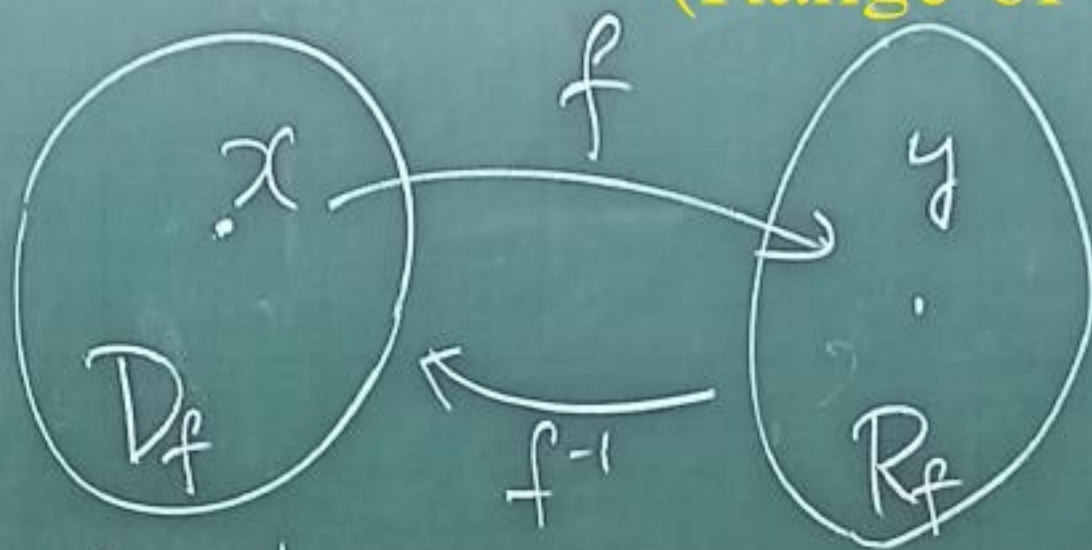


$f^{-1}(y_0)$ is not defined.

(Domain of f)

If $f: D_f \longrightarrow R_f$

(Range of f)



Then " $f^{-1}: R_f \longrightarrow D_f$ exists"

$\iff \begin{cases} f \text{ is one to one on } D_f \\ f \text{ maps } D_f \text{ onto } R_f \end{cases}$

One to one: $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$

onto: $R_f = \{f(x), x \in D_f\}$

Proposition $f: D_f \rightarrow R_f$

If f^{-1} exists, then

(a): $f^{-1}(f(x)) = x$ for any $x \in D_f$

(b) $f(f^{-1}(y)) = y$ for any $y \in R_f$

Ex 2: The inverse function of $y = a^x$, $a > 0$
is $x = \log_a y$ (textbook: $y = \log_a x$)

$$\therefore \log_a(a^x) = x, \quad x \in \mathbb{R}$$

$$a^{\log_a y} = y, \quad y \in \mathbb{R}^+$$

Notation Difference

Original function, inverse function

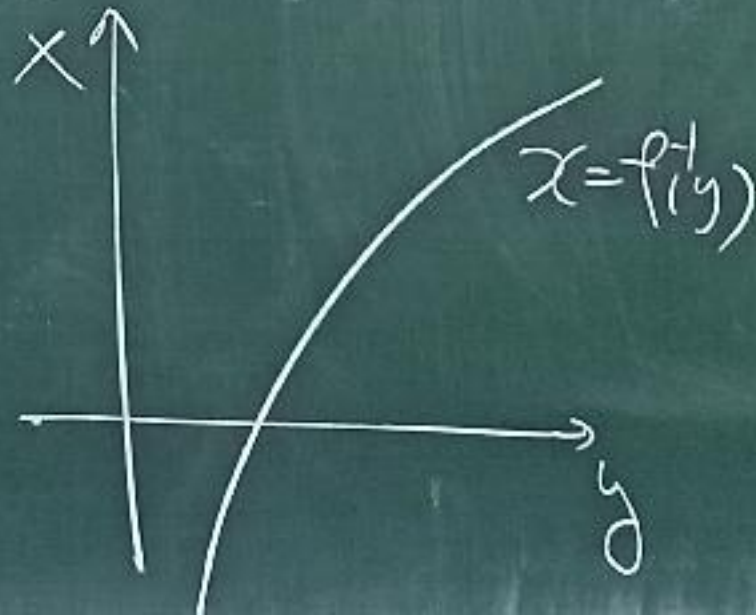
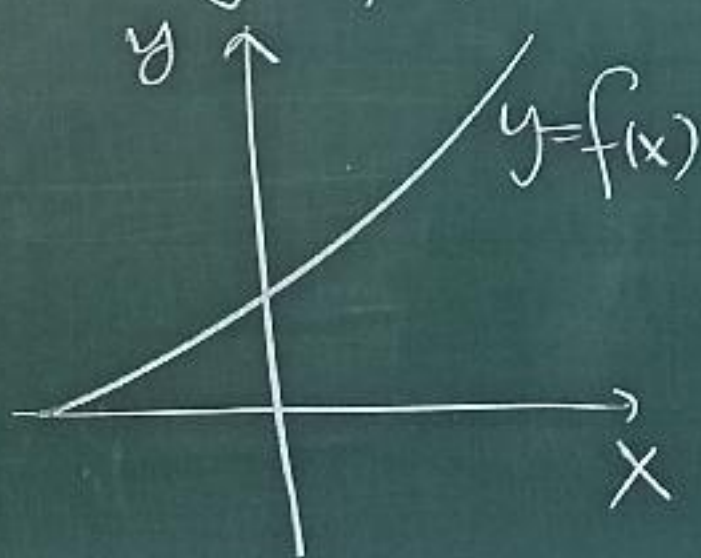
$$y = f(x) \quad y = f^{-1}(x)$$

textbook



$$y = f(x) \quad , \quad x = f^{-1}(y)$$

Here



Derivative of f^{-1}

$$f^{-1}(f(x)) = x \quad x \in D_f$$

$$\frac{d}{dx} \Rightarrow \frac{d}{dy} f^{-1}(y) \quad \cdot \quad f'(x) = 1$$

$|_{y=f(x)}$

$$\therefore \frac{d}{dy} f^{-1}(y) \Big|_{y=f(x)} = \frac{1}{f'(x)}$$

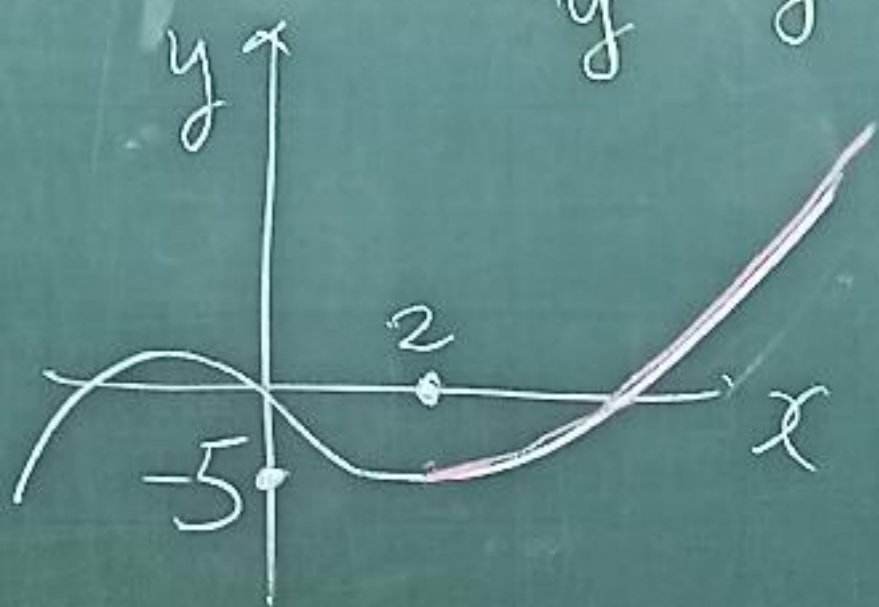
$$\text{or } \frac{d}{dy} f^{-1}(y) = \frac{1}{f'(x) \Big|_{x=f^{-1}(y)}}$$

Textbook:

$$\frac{d f^{-1}(x)}{dx} \Big|_{x=b} = \frac{1}{\frac{d f(x)}{dx} \Big|_{x=f^{-1}(b)}}$$

Eg8 $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$

Find $\frac{d}{dy} f^{-1}(y)$ at $x = -1 = f(3)$



Ans $\frac{d}{dy} f^{-1}(y) \Big|_{y=f(3)}$

$= \frac{1}{f'(x) \Big|_{x=3}} \left(\neq \frac{1}{f'(-1)} \right)$

$= \frac{1}{3x^2 - 6x} \Big|_{x=3} = \frac{1}{9}$

Eg: The inverse function
of $y = a^x$ is $x = \log_a y$

In particular, when
 $a = e$ (Euler number)

$$\log_e(\cdot) = \ln(\cdot)$$

(natural log)

$$\ln e^x = x, \quad x \in \mathbb{R}$$

$$e^{\ln y} = y, \quad y > 0$$

Note: The inverse function
of $y = e^x$ is $y = \ln x$
(text book notation)

$$\frac{d}{dx} \ln x = \frac{1}{\frac{d}{dx} e^x} = \frac{1}{e^x}$$

(WRONG)

Current notation: $y = e^x$
 $x = \ln y$

$$\frac{d}{dy} \ln y = \frac{1}{\frac{d}{dx} e^x} \Big|_{x=\ln y}$$
$$= \frac{1}{e^x} \Big|_{x=\ln y} = \frac{1}{e^{\ln y}} = \boxed{\frac{1}{y}} \quad \underline{\underline{\text{correct}}}$$