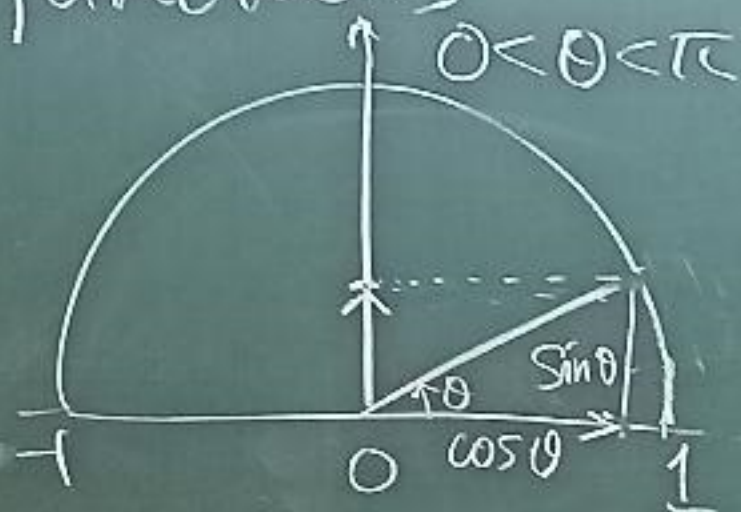


Derivatives of Trigonometric functions



$\sin \theta = \text{projection on } y \text{ axis}$
 $(\cos \theta)$ (x)

$$\tan \theta = \sin \theta / \cos \theta$$

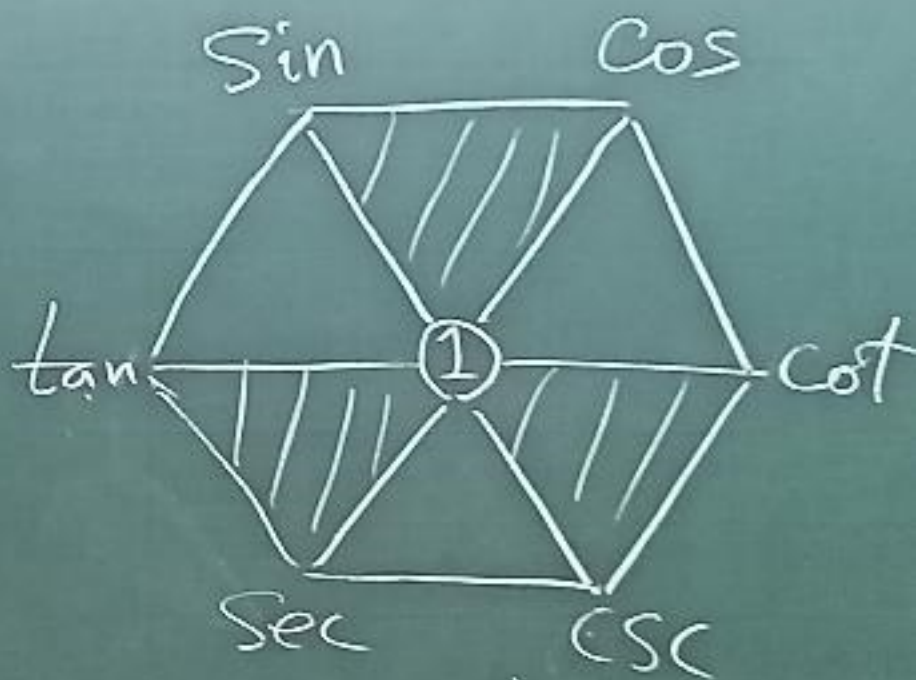
$$\cot \theta = \cos \theta / \sin \theta$$

$$\sec \theta = 1 / \cos \theta$$

$$\csc \theta = 1 / \sin \theta$$

When $-\pi < \theta < 0$

$$\sin \theta = -\sin(-\theta), \quad \cos \theta = \cos(-\theta)$$



$$a^2 + b^2 = c^2$$

We also have

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\text{Eg 1: } \cos(2A) = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$$

$$\sin(\theta \pm \pi) = -\sin \theta$$

$$\cos(\theta \pm \pi) = -\cos \theta$$

$$\text{Eg 2: } f(x) = \sin x, \quad f'(x) = ?$$

$$\begin{aligned} \text{Sol: } f'(x) &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h \rightarrow 0} \left(\sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \frac{\sin h}{h} \right) \\ &= \sin x \lim_{h \rightarrow 0} \frac{-2\sin^2(\frac{h}{2})}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} = \cos x \end{aligned}$$

Similarly, we have $\frac{d}{dx} \cos x = -\sin x$

Therefore (Memorize)

$$(\tan x)' = \left(\frac{\sin x}{\cos x} \right)' = \frac{\cos x (\sin x)' - \sin x (\cos x)'}{\cos^2 x} = \sec^2 x$$

$$(\cot x)' = \left(\frac{\cos x}{\sin x} \right)' = \dots = -\csc^2 x$$

$$(\sec x)' = \left(\frac{1}{\cos x} \right)' = \frac{\cos x \cdot 1' - 1(\cos x)'}{\cos^2 x} = \tan x \sec x$$

$$(\csc x)' = \left(\frac{1}{\sin x} \right)' = \dots = -\cot x \csc x$$

Ex 3. $\frac{d}{dx} \left(\frac{\sin x}{x} \right) = ?$

$$\text{Ans} = \frac{x(\sin x)' - (\sin x)x'}{x^2} = \frac{x \cos x - \sin x}{x^2}$$

Question: $\lim_{x \rightarrow 0} \frac{d}{dx} \left(\frac{\sin x}{x} \right) = ?$ (difficult!)

Chain Rule: $\frac{d}{dx} f(g(x)) = ?$

Thm If $y = f(u)$ is
differentiable at $u = g(x)$
and $g(x)$ is diff. at x ,
then $y = f \circ g(x) = f(g(x))$
is differentiable at x

$$\text{and } \frac{dy}{dx} = \frac{dy}{du} \Big|_{u=g(x)} \cdot \frac{du}{dx}$$

$$\left(\text{i.e. } \frac{d}{dx} f(g(x)) = \frac{df(u)}{du} \Big|_{u=g(x)} \cdot \frac{d}{dx} g(x) \right)$$

Eg 4 $\frac{d}{dx} \sin(\tan x)$

$$= \left. \frac{d \sin u}{du} \right|_{u=\tan x} \cdot \frac{d}{dx} \tan x$$

$$= \cos(\tan x) \cdot \sec^2 x$$

Eg 5. $(\cot^7 x)' = ?$

$$\text{Ans} = \left. \frac{du^7}{du} \right|_{u=\cot x} \cdot (\cot x)'$$

$$= 7 \cot^6 x \cdot (-\csc^2 x)$$

(i.e. $f(u) = u^7$, $g(x) = \cot x$
 $\cot^7 x = f(g(x))$)

$$\begin{aligned}
 \text{Eq 6 } \frac{d}{dx} \sin(x^2 + e^x) \\
 &= \cos(x^2 + e^x) \cdot (x^2 + e^x)' \\
 &= (2x + e^x) \cdot \cos(x^2 + e^x)
 \end{aligned}$$

$$\begin{aligned}
 \text{Eq 7 } \frac{d}{dx} \sin(\cos(x^2 + 1)) \\
 &= \cos(\cos(x^2 + 1)) \cdot (\cos(x^2 + 1))' \\
 &= \cos(\cos(x^2 + 1)) \cdot (-\sin(x^2 + 1) \cdot (x^2 + 1)') \\
 &= -\cos(\cos(x^2 + 1)) \cdot \sin(x^2 + 1) \cdot 2x
 \end{aligned}$$

Remark.

$$* \frac{d}{dx} (u(x))^n = n u^{n-1}(x) \cdot u'(x)$$

$$* \frac{d}{dx} e^{u(x)} = e^{u(x)} \cdot u'(x)$$

$$* \frac{d}{dx} \sin(u(x)) = \cos(u(x)) \cdot u'(x)$$

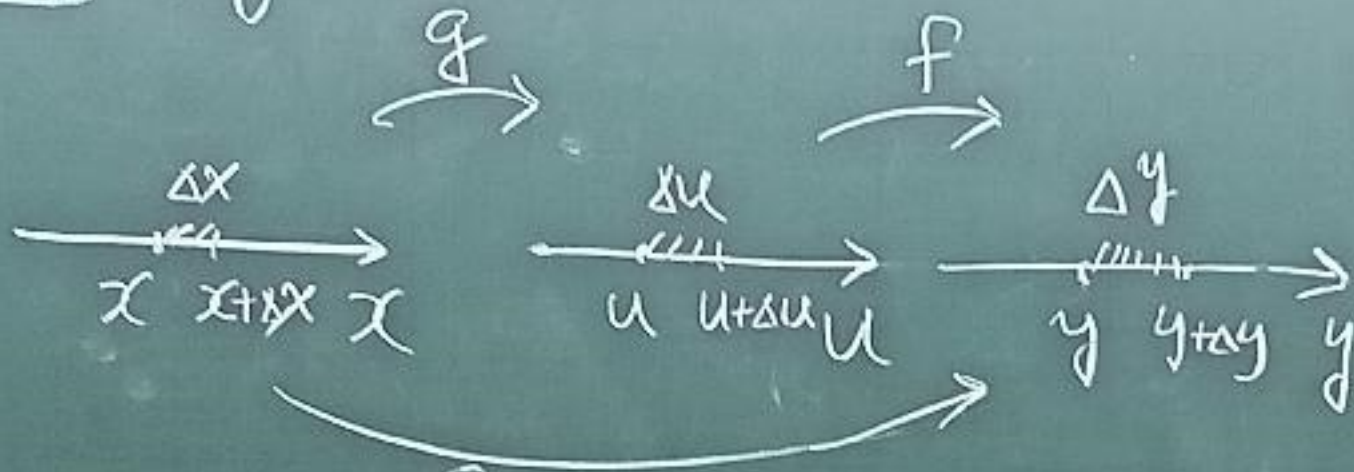
etc.

$$\begin{aligned} \text{Eg 7. } \frac{d}{dx} \left(\frac{1}{u(x)} \right) &= \frac{d}{dx} \left((u(x))^{-1} \right) \\ &= - (u(x))^{-2} u'(x) = \frac{-u'(x)}{u^2(x)} \end{aligned}$$

$$\begin{aligned} \text{Eg 8: } \frac{d}{dx} \sqrt{1+x^2} &= \frac{d}{dx} (1+x^2)^{\frac{1}{2}} \\ &= \frac{1}{2} (1+x^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}} \end{aligned}$$

$$\text{Eg 9: } \frac{d}{dx} e^{x^e} = e^{x^e} \cdot (x^e)' = e^{x^e} \cdot e x^{e-1}$$

pf (of Chain Rule)



$$g'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$f'(u) = \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \rightarrow 0} \frac{f(\underline{u+\Delta u}) - f(u)}{\Delta u}$$

Note: $\Delta u = g(x+\Delta x) - g(x)$

$$\therefore \underline{g(x+\Delta x)} = g(x) + \Delta u = \underline{u + \Delta u}$$

$$\frac{d}{dx} f(g(x)) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}$$

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta u} \right) \cdot \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta u}{\Delta x} \right)$$

Since g is diff. at x

$\Rightarrow g$ is cont. at x

i.e., " $\Delta x \rightarrow 0 \Rightarrow \Delta u \rightarrow 0$ "

$$\therefore \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta u} = \lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u}$$

$$\therefore f(g(x))' = \lim_{\Delta x \rightarrow 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$= \left(\lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} \right) \cdot \left(\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} \right)$$

$$= f'(u)|_{u=g(x)} \cdot g'(x)$$

$$\therefore f(g(x))' = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx} f(g(x)) = \frac{d}{du} f(u) \Big|_{u=g(x)} \cdot \frac{d}{dx} g(x)$$

Remark: $\Delta u = g(x + \Delta x) - g(x)$

it can happen that

$\Delta x \neq 0$, but $\Delta u = 0$,

this is a ~~minor~~ **minor** problem in

the proof. See section 3.11 for a correct and complete proof.