

Differentiation Rules

Ex 1 $f(x) = 1$

$$\frac{d}{dx} f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{1-1}{h} = 0$$

Ex 2 $f(x) = x^3$ $f'(x) = ?$

Ans: $f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= 3x^2 \end{aligned}$$

In general, if $f(x) = x^n$, $n \in \mathbb{N}$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\cancel{x^n} + n x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots - \cancel{x^n}}{h}$$

$$= n x^{n-1} (+0 + 0 + \dots)$$

$$\text{Ex 3: } f(x) = x^{\frac{1}{3}}, x \neq 0$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}$$

$$(a-b)(a^2+ab+b^2) = a^3-b^3$$

$$= \lim_{h \rightarrow 0} \frac{((x+h)^{\frac{1}{3}} - x^{\frac{1}{3}})((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}})}{h((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}})}$$

$$= \lim_{h \rightarrow 0} \frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}$$

$$= \frac{1}{3} \frac{1}{x^{\frac{2}{3}}} = \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3} x^{\frac{1}{3}-1}$$

Remark: In fact

$$\frac{d}{dx} x^n = n x^{n-1} \text{ holds for } n \in \mathbb{Z} \text{ (integers)}$$

$$n = \frac{p}{q}, p, q \in \mathbb{N}, p \neq 0$$

and also for $n \in \mathbb{R}$ (later)

Derivative Rules

$$\frac{d}{dx} (C u(x)) = C \frac{d}{dx} u(x) \dots (1)$$

$C = \text{constant}$

$$\frac{d}{dx} (u(x) \pm v(x)) = \frac{du}{dx} \pm \frac{dv}{dx} \dots (2)$$

$$\frac{d}{dx} (u(x) \cdot v(x)) = u'v + uv' \dots (3)$$

$$\frac{d}{dx} \left(\frac{u}{v} \right) = \frac{v u' - u v'}{v^2} \dots (4)$$

$(v(x) \neq 0)$

pf of (3):

$$(uv)' = \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

$$= \lim_{h \rightarrow 0} u(x+h) \frac{v(x+h) - v(x)}{h} + \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} v(x)$$

$$= u(x) v'(x) + u'(x) v(x)$$

pf of (4):

$$\frac{u}{v} = u \cdot \left(\frac{1}{v}\right), \quad v(x) \neq 0$$

From (3), it remains to evaluate

$$\begin{aligned} \left(\frac{1}{v}\right)'(x) &= \lim_{h \rightarrow 0} \frac{\frac{1}{v(x+h)} - \frac{1}{v(x)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{v(x) - v(x+h)}{v(x+h)v(x)}}{h} \\ &= - \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} \cdot \frac{1}{v(x+h)v(x)} \\ &= - \frac{v'(x)}{v^2(x)} \Rightarrow (4) \end{aligned}$$

Ex4: $f(x) = x^4 - 2x^2 + 2$

Find all horizontal tangents of $y = f(x)$.

Ans Solve for x from $f'(x) = 0$

$$f'(x) = 4x^3 - 4x + 0$$

$$f'(x) = 0 \iff x = 0, \pm 1$$

horizontal tangents at $(0, f(0))$

$(1, f(1))$ and $(-1, f(-1))$ are

$$\frac{y-2}{x-0} = 0, \quad \frac{y-1}{x-1} = 0 \quad \text{and} \quad \frac{y-1}{x+1} = 0$$

Exponential functions

$$y = a^x, \quad x \in \mathbb{R}, \quad a > 0$$



$$\frac{d}{dx} a^x = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \left(\lim_{h \rightarrow 0} \frac{a^h - 1}{h} \right)$$

$$= g(a) a^x$$

$g(a)$

Here $g(a) = \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$
(if the limit exists)

$$g(1) = \lim_{h \rightarrow 0} \frac{0}{h} = 0.$$

$$g(a^2) = \lim_{h \rightarrow 0} \frac{a^{2h} - 1}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{a^h - 1}{h} \right) (a^h + 1) = 2g(a)$$

$$a > 1 \Rightarrow g(a) \geq 0 \quad \left(\begin{array}{l} \text{In fact} \\ > 0 \end{array} \right)$$

$$0 < a < 1 \Rightarrow g(a) < 0$$



Define the Euler number "e"

satisfying $f(e) = 1$

i.e. $\lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$, $e \approx 2.71828 \dots$

or $\frac{d}{dx} e^x = e^x$

Ex 5: $f(x) = e^{-x}$, $f'(x) = ?$

Ans: $\frac{d}{dx} f(x) = \frac{d}{dx} \left(\frac{1}{e^x} \right)$

from (4)

$$\Rightarrow f'(x) = \frac{e^x \cdot 1' - 1(e^x)'}{(e^x)^2}$$

$$= -e^{-x}$$