Recall: $\lim_{x \to c} f(x) = L$

For every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that

"$0 < |x-c| < \delta \Rightarrow |f(x) - L| < \varepsilon$"

(x is sufficiently close to c)

<u>Def</u>: $\lim_{x \to \pm\infty} f(x) = L$

For every $\varepsilon > 0$, there exists a corresponding $M \in \mathbb{R}$ such that

N

"$x > M \Longrightarrow |f(x) - L| < \varepsilon$"

$x < N$

(x is sufficiently close to $\pm\infty$)

Def: $\lim_{x \to c, \underline{c^+}, \underline{c^-}} f(x) = \begin{matrix} +\infty \\ (-\infty) \end{matrix}$

($|f(x) - L| < \varepsilon$ means "$f(x)$ is close to L"
Replace it by "$f(x)$ is close to $\begin{matrix} +\infty \\ (-\infty) \end{matrix}$")

For every $\underline{\underline{B > 0}}_{(1)}$ there exists a corresponding $\delta > 0$, such that

" $0 < |x - c| < \delta \Rightarrow f(x) > B$ "

$\underline{0 < x - c < \delta}$ $\qquad \underline{\underline{(< -B)}}_{(2)}$

$\underline{-\delta < x - c < 0}$

(or (1) $\to B \in \mathbb{R}$. (2) $\to \begin{matrix} > B \\ < B \end{matrix}, < -B$)

Eg1 $\lim\limits_{x\to 0^{\pm}} 2^{\frac{1}{x}} = \begin{matrix}+\infty\\ 0\end{matrix}$

Eg2 $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 0$

Eg3. $\lim\limits_{x\to\infty} x\sin\frac{1}{x}$

does not exist. (oscillatory)

Eg4 $\lim\limits_{\theta\to\frac{\pi}{2}^+} \tan\theta = \lim\limits_{\theta\to\frac{\pi}{2}^+} \frac{\sin\theta}{\cos\theta}$

$= -\infty$

Eg 4. Prove that $\lim\limits_{x \to \infty} \frac{1}{x} = 0$

Pf For every $\varepsilon > 0$

$$\left| \frac{1}{x} - 0 \right| < \varepsilon$$

$$\Leftrightarrow -\varepsilon < \frac{1}{x} < \varepsilon$$

$$\Leftrightarrow \begin{cases} 0 < \frac{1}{x} < \varepsilon \\ \text{or } -\varepsilon < \frac{1}{x} < 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x > \frac{1}{\varepsilon} \\ \text{or } x < \frac{-1}{\varepsilon} \end{cases}$$

$$\Leftarrow x > M$$

(take $M = \frac{1}{\varepsilon}$ will do)

The most important sentence in the proof

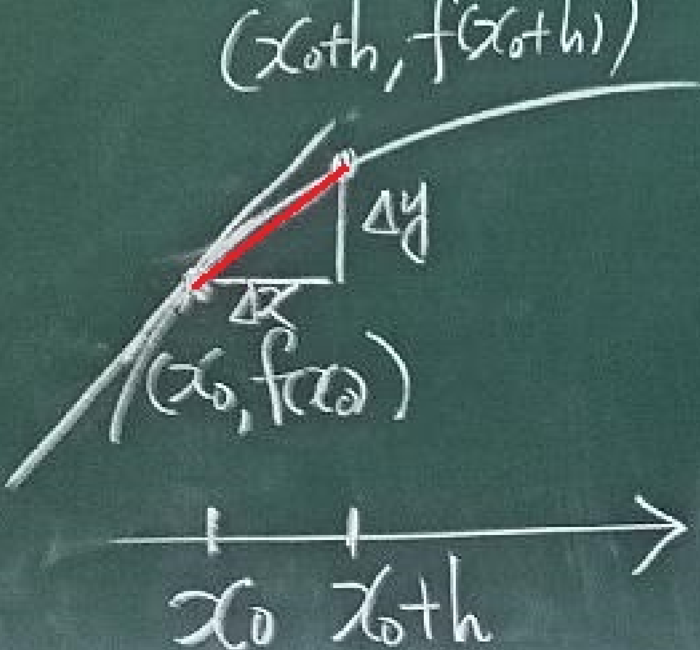
Chap 3.

Def Derivative of f at x_0

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$$\left(= \lim \left(\text{slope of secant (割線)} \right) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \right.$$

$= \text{slope of tangent (切线)}$



$y = f(x)$

(When x = time
$f'(x_0)$ = rate of change of y at x_0)

$f'(x_0)$ = derivative of f at x_0

$f'(x)$ = derivative of f as a function of x

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Notation: $\frac{d}{dx} f(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$, $D_x f$, $f'(x)$

$$\frac{d}{dx} f(x_0) = \frac{d}{dx} f(x) \Big|_{x=x_0} = \frac{dy}{dx} \Big|_{x=x_0}$$

Eg5. $f(x)=\frac{1}{x},\ x\neq 0$

$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x-(x+h)}{(x+h)x}}{h}$$

$$=\lim_{h\to 0}\frac{-1}{(x+h)x}=\frac{-1}{x^2}$$

Eg 6: $f(x)=\sqrt{x}$, $x \geq 0$

$f'(x) = ?$

Ans: if $x>0$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$=\lim_{h\to 0}\frac{(\sqrt{x+h}+\sqrt{x})(\sqrt{x+h}-\sqrt{x})}{(\sqrt{x+h}+\sqrt{x})\ h}$$

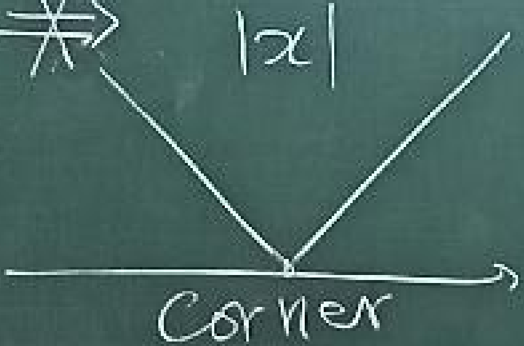
$$=\lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{2}\frac{1}{\sqrt{x}}$$

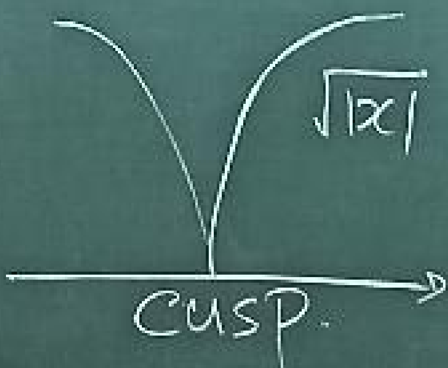
If $x=0$ $\lim_{h\to 0^+}\frac{\sqrt{h+0}-\sqrt{0}}{h}=\lim_{h\to 0^+}\frac{1}{\sqrt{h}}=+\infty$

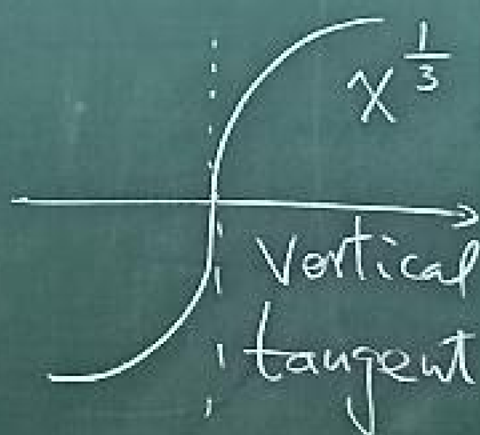
Relation between Continuity and differentiability.

$f(x)$ is cont. at x_0 $\overset{\not\Longrightarrow}{\Longleftarrow}$ $f(x)$ is diff. at x_0

"$\not\Longrightarrow$"







"$\Longleftarrow$" <u>Thm</u> If f has a derivative at c then f is continuous at c.

pf f is differentiable at c

$$\Leftrightarrow \lim_{x\to c}\frac{f(x)-f(c)}{x-c} \text{ exists } (=f'(c)) \quad (1)$$

f is continuous at c

$$\Leftrightarrow \lim_{x\to c}\left(f(x)-f(c)\right)=0 \quad (2)$$

$$(1)\Rightarrow \lim_{x\to c} f(x)-f(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}(x-c)$$

$$=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot\lim_{x\to c}(x-c)=f'(c)\cdot 0=0$$

$\#$

Differentiation Rules

Eg1 $f(x) = 1$

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1-1}{h} = 0$$

Eg2. $f(x) = x^3$. $f'(x) = ?$

Ans: $f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$.

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$= 3x^2$$

In general if $f(x)=x^n$, $n\in\mathbb{N}$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^n}+nx^{n-1}h+\binom{n}{2}x^{n-2}h^2+\cdots-\cancel{x^n}}{h}$$

$$=nx^{n-1}(+0+0+\cdots)$$

Eg 3: $f(x) = x^{\frac{1}{3}},\ x \neq 0$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}$$

$$(a-b)(a^2+ab+b^2) = a^3 - b^3$$

$$= \lim_{h \to 0} \frac{\left((x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}\right)\left((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}{h\left((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}$$

$$= \lim_{h \to 0} \frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}$$

$$= \frac{1}{3} \frac{1}{x^{\frac{2}{3}}} = \frac{1}{3} x^{\frac{-2}{3}} = \frac{1}{3} x^{\frac{1}{3}-1}$$

Remark: In fact

$\frac{d}{dx}x^n = nx^{n-1}$ holds for

$n = \frac{q}{p}$, $p, q \in \cancel{\mathbb{N}}$ $\mathbb{Z}$ (integers), $p \neq 0$

and also for $n \in \mathbb{R}$ (later)

Derivative Rules

$$\frac{d}{dx}(c\,u(x)) = c\frac{d}{dx}u(x) \cdots (1)$$

c = constant

$$\frac{d}{dx}(u(x) \pm v(x)) = \frac{du}{dx} \pm \frac{dv}{dx} \cdots (2)$$

$$\frac{d}{dx}(u(x) \cdot v(x)) = u'v + uv' \cdots (3)$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \cdots (4)$$

$(v(x) \neq 0)$

pf of (3):

$$(uv)' = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$

$$= \lim_{h\to 0} u(x+h)\frac{v(x+h)-v(x)}{h} + \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} v(x)$$

$$= u(x)v'(x) + u'(x)v(x)$$

Pf of (4):

$$\frac{u}{v} = u \cdot \left(\frac{1}{v}\right), \quad v(x) \neq 0$$

From (3), it remains to evaluate

$$\left(\frac{1}{v}\right)'(x) = \lim_{h\to 0} \frac{\frac{1}{v(x+h)} - \frac{1}{v(x)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{v(x) - v(x+h)}{v(x+h)\,v(x)}}{h}$$

$$= -\lim_{h\to 0} \frac{v(x+h) - v(x)}{h} \cdot \frac{1}{v(x+h)\,v(x)}$$

$$= -\frac{v'}{v^2}(x) \Longrightarrow (4)$$

Eg4. $f(x) = x^4 - 2x^2 + 2$

Find all horizontal tangents of $y = f(x)$.

Ans. Solve for x from $f'(x) = 0$

$$f'(x) = 4x^3 - 4x + 0$$

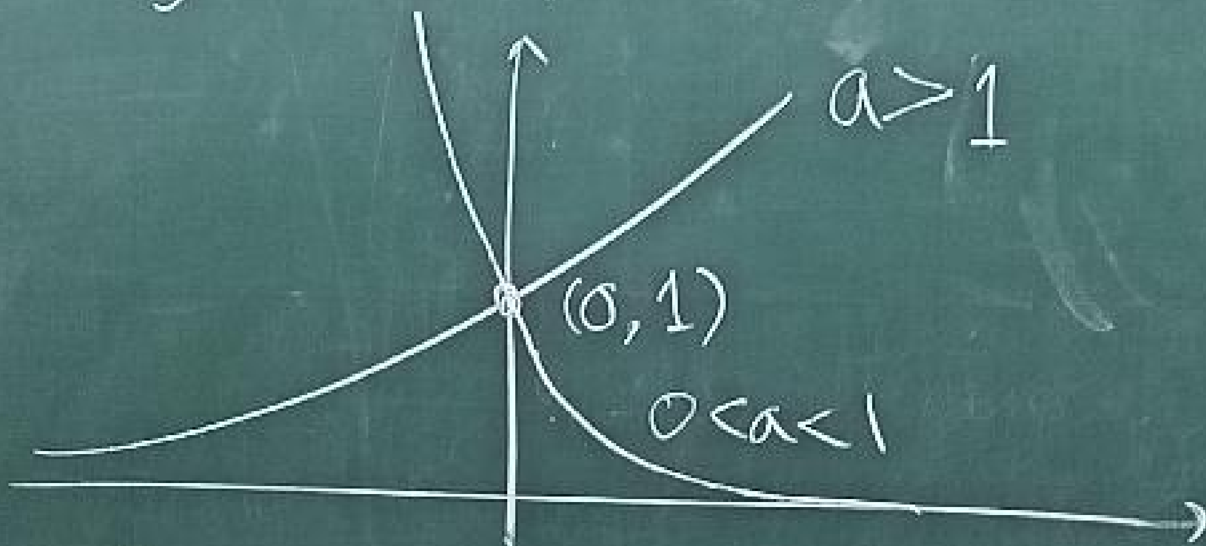
$$f'(x) = 0 \iff x = 0, \pm 1$$

horizontal tangents at $(0, f(0))$ $(1, f(1))$ and $(-1, f(-1))$ are

$$\frac{y-2}{x-0} = 0, \quad \frac{y-1}{x-1} = 0 \text{ and } \frac{y-1}{x+1} = 0$$

Exponential functions

$$y = a^x, \; x \in \mathbb{R}, \; a > 0$$



$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$= a^x\left(\underbrace{\lim_{h\to 0}\frac{a^h-1}{h}}_{g(a)}\right)$$

$$= g(a)\,a^x$$

Here $g(a) = \lim_{h \to 0} \frac{a^h - 1}{h}$

(if the limit exists)

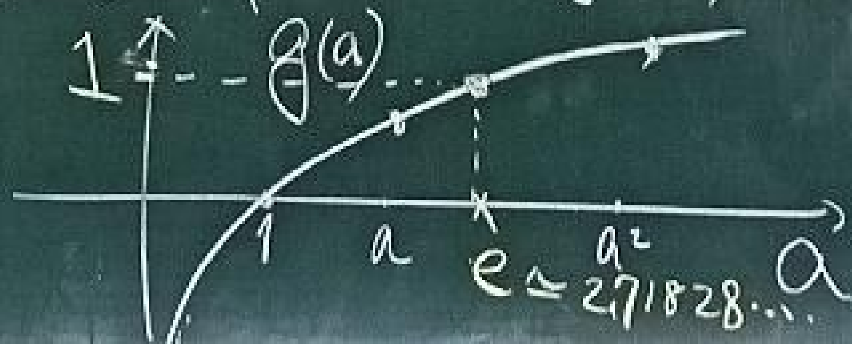
$$g(1) = \lim_{h \to 0} \frac{0}{h} = 0.$$

$$g(a^2) = \lim_{h \to 0} \frac{a^{2h} - 1}{h}$$

$$= \lim_{h \to 0} \left(\frac{a^h - 1}{h} \right) (a^h + 1) = 2g(a)$$

$a > 1 \Rightarrow g(a) \geq 0$ (In fact > 0)

$0 < a < 1 \Rightarrow g(a) < 0$



Define the Euler number "e"

satisfying $g(e) = 1$

i.e. $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$, $e \simeq 2.71828\ldots$

or $\frac{d}{dx} e^x = e^x$

Eg 5: $f(x) = e^{-x}$, $f'(x) = ?$

Ans: $\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{1}{e^x}\right)$

from (4)

$$\Rightarrow f'(x) = \frac{e^x \cdot 1' - 1(e^x)'}{(e^x)^2}$$

$$= -e^{-x}$$