Recall: $\lim_{x \to c} f(x) = L$

For every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that "$\underline{0 < |x - c| < \delta} \Rightarrow |f(x) - L| < \varepsilon$"

(x is sufficiently close to c)

<u>Def</u>: $\lim_{x \to \pm\infty} f(x) = L$

For every $\varepsilon > 0$, there exists a corresponding $M \in \mathbb{R}$ N such that

" $x > M$ $\Longrightarrow |f(x) - L| < \varepsilon$"

$x < N$

(x is sufficiently close to $\pm\infty$)

Def: $\lim_{x \to c, \underline{c^+}, \underline{c^-}} f(x) = \begin{matrix} +\infty \\ (-\infty) \end{matrix}$

($|f(x) - L| < \varepsilon$ means "$f(x)$ is close to L"

Replace it by "$f(x)$ is close to $\begin{matrix} +\infty \\ (-\infty) \end{matrix}$")

For every $\underline{\underline{B > 0}}_{(1)}$ there exists a corresponding $\delta > 0$, such that

" $0 < |x - c| < \delta \Longrightarrow f(x) > B$ "

$\underline{0 < x - c < \delta}$ $\qquad \underline{\underline{(< -B)}}_{(2)}$

$\underline{-\delta < x - c < 0}$

(or (1) $\to B \in \mathbb{R}$. (2) $\to$ $\begin{matrix} > B \\ < B \end{matrix}$, $< -B$)

Eg1 $\lim\limits_{x \to 0^{\pm}} 2^{\frac{1}{x}} = \begin{matrix} +\infty \\ 0 \end{matrix}$

Eg2 $\lim\limits_{x \to \infty} \frac{\sin x}{x} = 0$

$0 <= |f(x)| <= 1/|x|$, apply SandwichThm

Eg3. $\lim\limits_{x \to \infty} x \sin x$

does not exist. (oscillatory)

Eg4 $\lim\limits_{\theta \to \frac{\pi}{2}^{+}} \tan\theta = \lim\limits_{\theta \to \frac{\pi}{2}^{+}} \frac{\sin\theta}{\cos\theta}$

$= -\infty$

Eg5 $\lim\limits_{x\to\infty} x \sin\frac{1}{x} = \lim\limits_{x\to\infty} \dfrac{\sin\frac{1}{x}}{\frac{1}{x}}$ $(t=\frac{1}{x})$

$= \lim\limits_{t\to 0^+} \dfrac{\sin t}{t} = 1$

Eg6 $\lim\limits_{x\to\infty} \left(x-\sqrt{x^2+6}\right)$

$= \lim\limits_{x\to\infty} \left(x-\sqrt{x^2+6}\right)\left(\dfrac{x+\sqrt{x^2+6}}{x+\sqrt{x^2+6}}\right)$

$= \lim\limits_{x\to\infty} \dfrac{x^2-\sqrt{x^2+6}^{\,2}}{x+\sqrt{x^2+6}} = \dfrac{-6}{\infty} = 0$

Eg 7 $\lim_{x\to\infty} \dfrac{2x^5-6x^4+1}{3x^2+x-7}$

$= \lim_{x\to\infty} \dfrac{2-6x^{-1}+x^{-5}}{3x^{-3}+x^{-4}-7x^{-5}}$

$= \dfrac{2}{0^+} = +\infty$

Eg 8. Prove that $\lim_{x \to \infty} \frac{1}{x} = 0$

Pf For every $\varepsilon > 0$

$$\left| \frac{1}{x} - 0 \right| < \varepsilon$$

$$\Leftrightarrow -\varepsilon < \frac{1}{x} < \varepsilon$$

$$\Leftrightarrow \begin{cases} 0 < \frac{1}{x} < \varepsilon \\ \text{or } -\varepsilon < \frac{1}{x} < 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x > \frac{1}{\varepsilon} \\ \text{or } x < \frac{-1}{\varepsilon} \end{cases}$$

$$\Leftarrow x > M$$

($\underline{\text{take } M = \frac{1}{\varepsilon}}$ will do)

The most important sentence in the proof

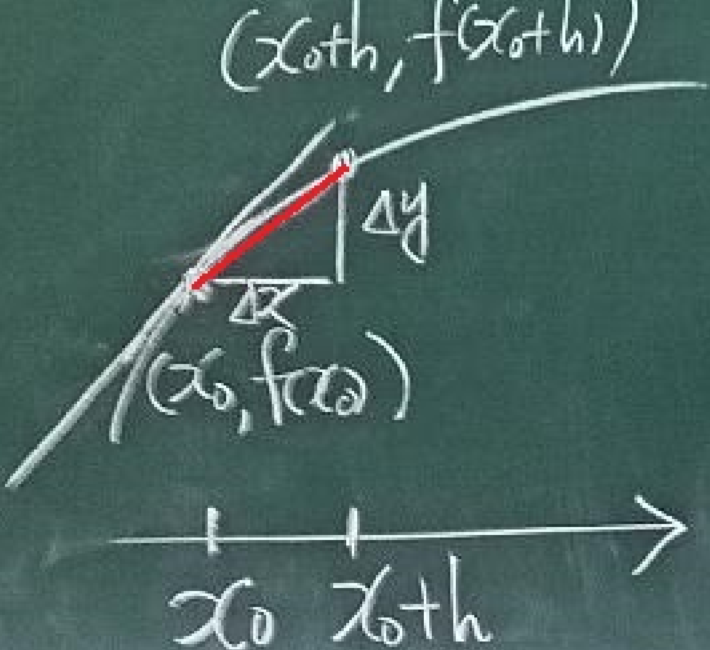
Chap 3.

Def Derivative of f at x_0

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

$\Big(= \lim \big(\text{slope of secant (割線)}\big) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

$= \text{slope of tangent (切线)} \Big)$



$y = f(x)$

(when x = time
$f'(x_0)$ = rate of
change of y at x_0)

$f'(x_0)$ = derivative of f at x_0

$f'(x)$ = derivative of f as a function of x

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Notation: $\frac{d}{dx} f(x)$, $\frac{df}{dx}$, $\frac{dy}{dx}$, $D_x f$, $f'(x)$

$$\frac{d}{dx} f(x_0) = \frac{d}{dx} f(x)\Big|_{x=x_0} = \frac{dy}{dx}\Big|_{x=x_0}$$

Eg1 $f(x)=\frac{1}{x},\ x\neq 0$

$$f'(x)=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$=\lim_{h\to 0}\frac{\frac{x-(x+h)}{(x+h)x}}{h}$$

$$=\lim_{h\to 0}\frac{-1}{(x+h)x}=\frac{-1}{x^2}$$

Eg2: $f(x)=\sqrt{x}$, $x \geq 0$

$f'(x) = ?$

<u>Ans</u>: if $x > 0$

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{x+h} + \sqrt{x})(\sqrt{x+h} - \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})\, h}$$

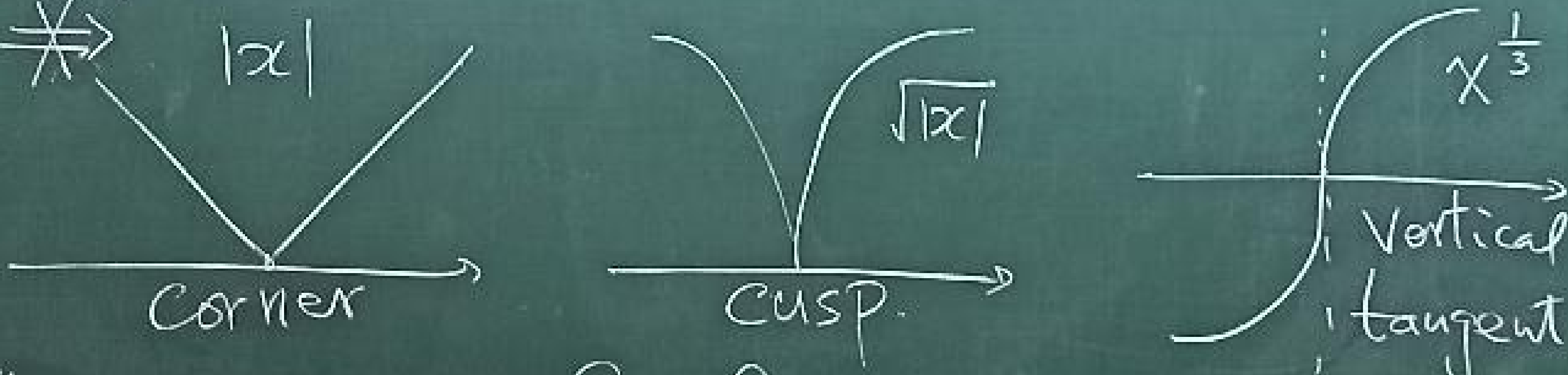
$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2} \frac{1}{\sqrt{x}}$$

If $x=0$ $\lim_{h \to 0^+} \frac{\sqrt{h+0} - \sqrt{0}}{h} = \lim_{h \to 0^+} \frac{1}{\sqrt{h}} = +\infty$

Relation between Continuity and differentiability.

$f(x)$ is cont. at x_0 $\overset{\not\Rightarrow}{\Leftarrow}$ $f(x)$ is diff. at x_0

" $\not\Rightarrow$ "



" $\Leftarrow$ " Thm If f has a derivative at c then f is continuous at c.

pf f is differentiable at c

$$\Leftrightarrow \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \text{ exists } (1)$$

$(= f'(c))$

---

f is continuous at c

$$\Leftrightarrow \lim_{x \to c} \left(f(x) - f(c) \right) = 0 \quad (2)$$

$$(1) \Rightarrow \lim_{x \to c} f(x) - f(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c} (x-c)$$

$$= \lim_{x \to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x \to c} (x-c) = f'(c) \cdot 0 = 0$$

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