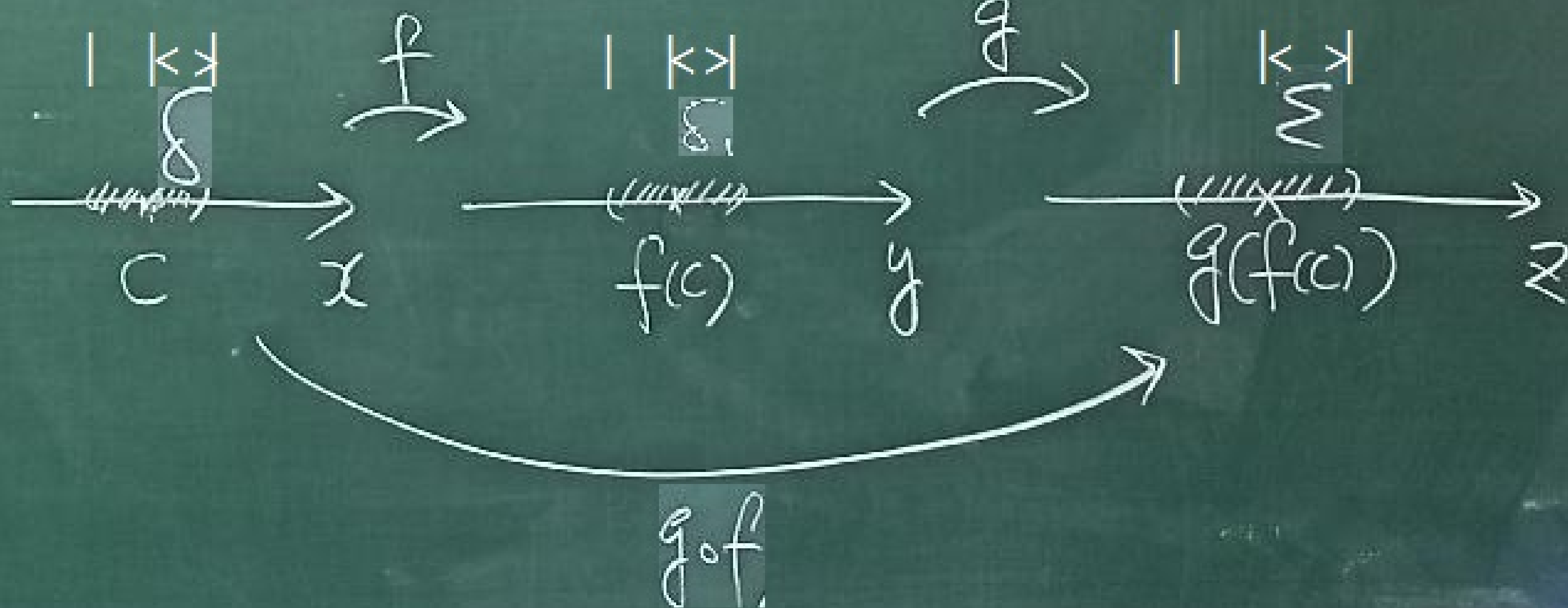
Thm 9 [Composite of continuous functions]

If f is continuous at c, g is cont. at $f(c)$. then $(g\circ f)(x) = g(f(x))$ is continuous at c.

i.e. $\lim_{x\to c} g(f(x)) = g(f(c))$



Pf: To show that $g \circ f$ is continuous, we need to show for any $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that

" $|x - c| < \delta \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$ "

To do this, we note that, for any $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that,

$(*_1)$ " $|y - f(c)| < \delta_1 \Rightarrow |g(y) - g(f(c))| < \varepsilon$ "

(g is continuous at $f(c)$)

For this $\delta_1 > 0$, there exists a corresponding $\delta > 0$ such that

$(*_2)$ "$|x-c|<\delta \Rightarrow |f(x)-f(c)|<\delta_1$"

(f is continuous at c)

$(*_1)+(*_2) \Longrightarrow$

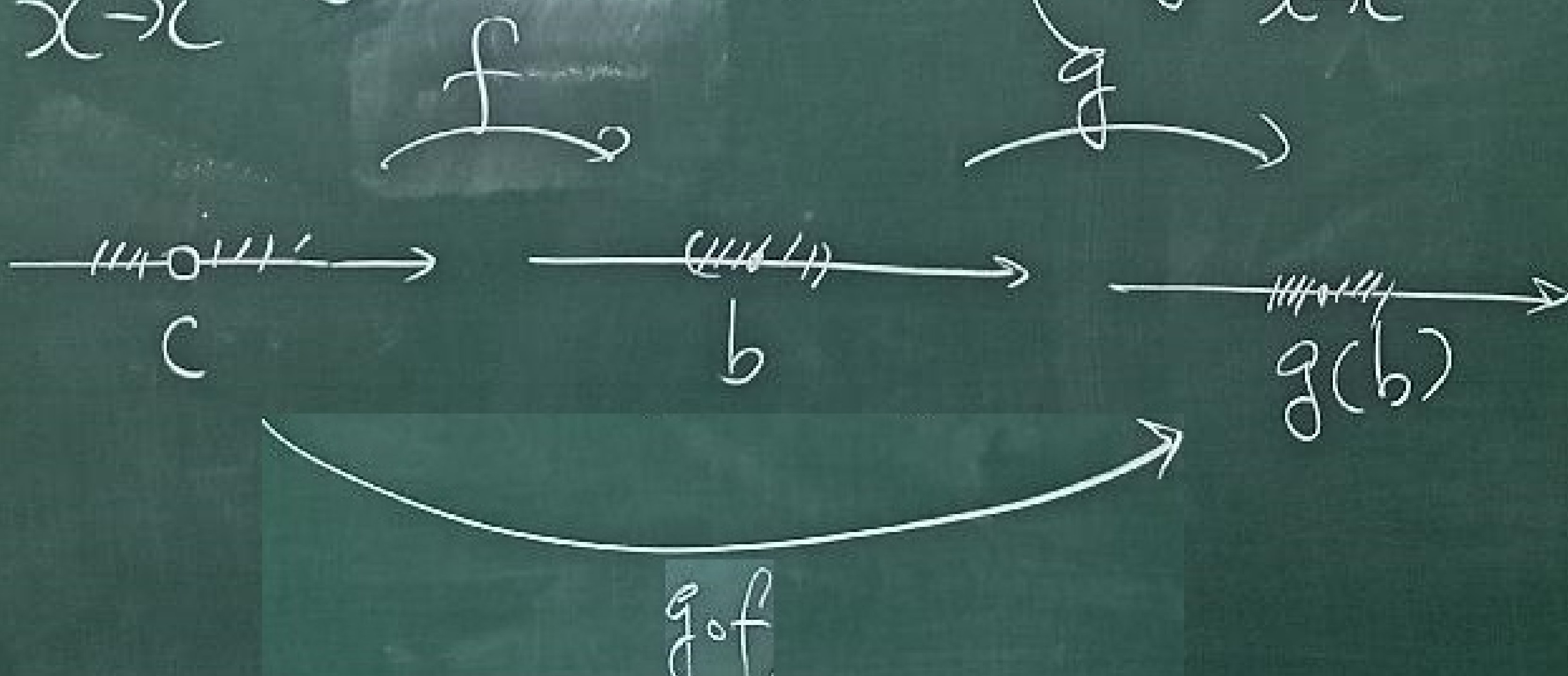
"$|x-c|<\delta \Rightarrow |f(x)-f(c)|<\delta_1 \Rightarrow |g(f(x))-g(f(c))|<\varepsilon$"

Eg1: $\lim_{x\to 0}\sqrt{x+1}\,2^{\sin x} = \sqrt{0+1}\cdot 2^{\sin 0} = 1$

Since both $\sqrt{x+1}$ and $2^{\sin x}$ are continuous function (Thm 9), so is $\sqrt{x+1}\,2^{\sin x}$ (Thm 8)

Thm 10 Similarly, if g is continuous at b and $\lim_{x \to c} f(x) = b$, then

$$\lim_{x \to c} g(f(x)) = g(b) \quad \left(= g\left(\lim_{x \to c} f(x) \right) \right)$$



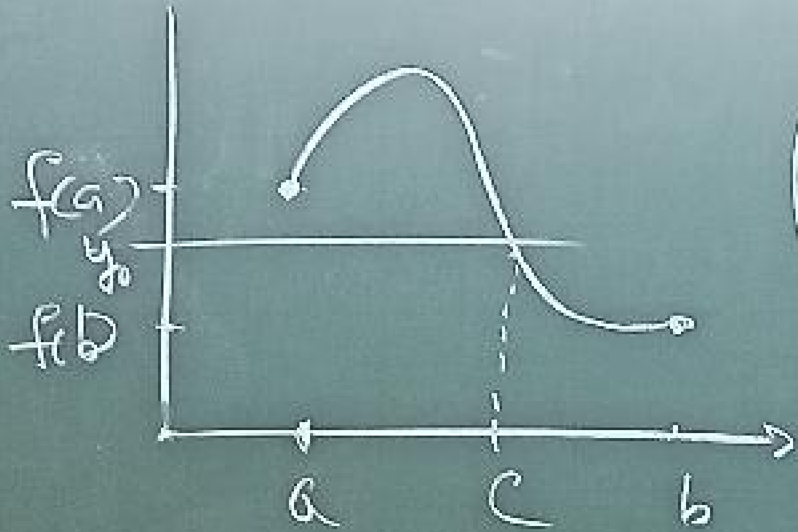
Thm 11 [Intermediate Value Thm]

f is continuous on $[a, b]$

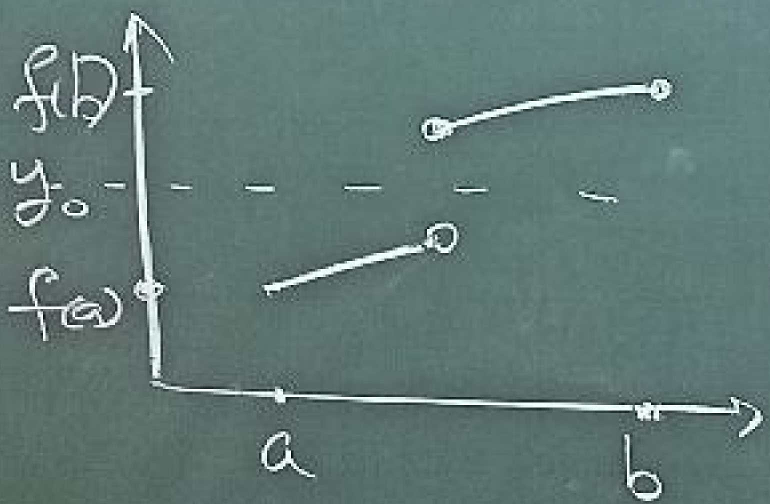
$\Longrightarrow f$ takes any value between $f(a)$ and $f(b)$.

i.e. if y_0 is any value between $f(a)$ and $f(b)$, then there exists a $c \in [a,b]$ such that

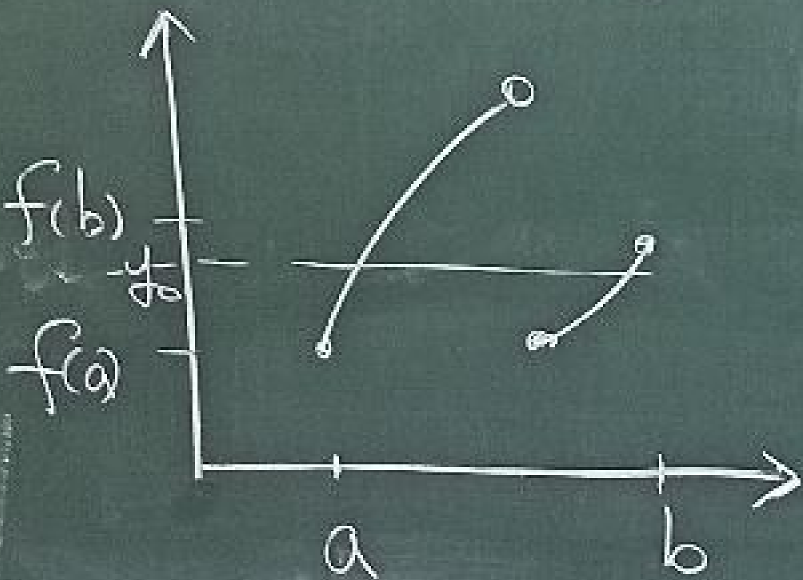
$$f(c) = y_0$$



(Typical case)



(I.V.T. may fail if f is not cont. on [a,b])



(f takes any value between f(a) and f(b) $\not\Rightarrow$ f is cont. on [a, b])

Eg 2. Show that $f(x) = x^3 - x - 1 = 0$ has a root in $(1,2)$

Sol. $f(1) = -1$, $f(2) = 5$.

$\xrightarrow{\text{I.V.T.}} \underset{(存在)}{\exists}\, x \in (1,2),\ f(x) = 0$.

Eg 3. Show that

$\sqrt{2x+5} = 4 - x^2$ has a solution.

Sol. Let $f(x) = \sqrt{2x+5} - (4 - x^2)$

$f(0) = \sqrt{5} - 4 < 0$, $f(2) = 3 - 0 > 0$

$\text{I.V.T.} \Rightarrow \exists\, x \in (0,2),\ f(x) = 0$, $\left(\sqrt{2x+5} = 4 - x^2\right)$

Recall: $\lim_{x \to c} f(x) = L$

For every $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that

"$\underline{0 < |x - c| < \delta} \Rightarrow |f(x) - L| < \varepsilon$"

(x is sufficiently close to c)

<u>Def</u>: $\lim_{x \to \pm\infty} f(x) = L$

For every $\varepsilon > 0$, there exists a corresponding $M \in \mathbb{R}$ such that

N

"$x > M \Longrightarrow |f(x) - L| < \varepsilon$"

$x < N$

(x is sufficiently close to $\pm\infty$)

Def: $\lim\limits_{x \to c, \underline{c^+}, \underline{c^-}} f(x) = \begin{matrix} +\infty \\ (-\infty) \end{matrix}$

($|f(x) - L| < \varepsilon$ means "$f(x)$ is close to L"

Replace it by "$f(x)$ is close to $\begin{matrix} +\infty \\ (-\infty) \end{matrix}$")

For every $\underline{\underline{B > 0}}_{(1)}$ there exists a corresponding $\delta > 0$, such that

" $0 < |x - c| < \delta \Rightarrow f(x) > B$ "

$\underline{0 < x - c < \delta}$ $\qquad \underline{\underline{(< -B)}}\ (2)$

$\underline{-\delta < x - c < 0}$

(or $(1) \to B \in \mathbb{R}$. $(2) \to \begin{matrix} > B \\ < B \end{matrix}, < -B$)

Eg1 $\lim_{x \to 0^{\pm}} 2^{\frac{1}{x}} = \begin{matrix} +\infty \\ 0 \end{matrix}$

Eg2 $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

Eg3. $\lim_{x \to \infty} x \sin \frac{1}{x}$

does not exist. (oscillatory)

Eg4 $\lim_{\theta \to \frac{\pi}{2}^{+}} \tan\theta = \lim_{\theta \to \frac{\pi}{2}^{+}} \frac{\sin\theta}{\cos\theta}$

$= -\infty$

Eg 4. Prove that $\lim_{x \to \infty} \frac{1}{x} = 0$

Pf For every $\varepsilon > 0$

$$\left| \frac{1}{x} - 0 \right| < \varepsilon$$

$$\Leftrightarrow -\varepsilon < \frac{1}{x} < \varepsilon$$

$$\Leftrightarrow \begin{cases} 0 < \frac{1}{x} < \varepsilon \\ \text{or } -\varepsilon < \frac{1}{x} < 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x > \frac{1}{\varepsilon} \\ \text{or } x < \frac{-1}{\varepsilon} \end{cases}$$

$$\Leftarrow x > M$$

(take $M = \frac{1}{\varepsilon}$ will do)

The most important sentence in the proof