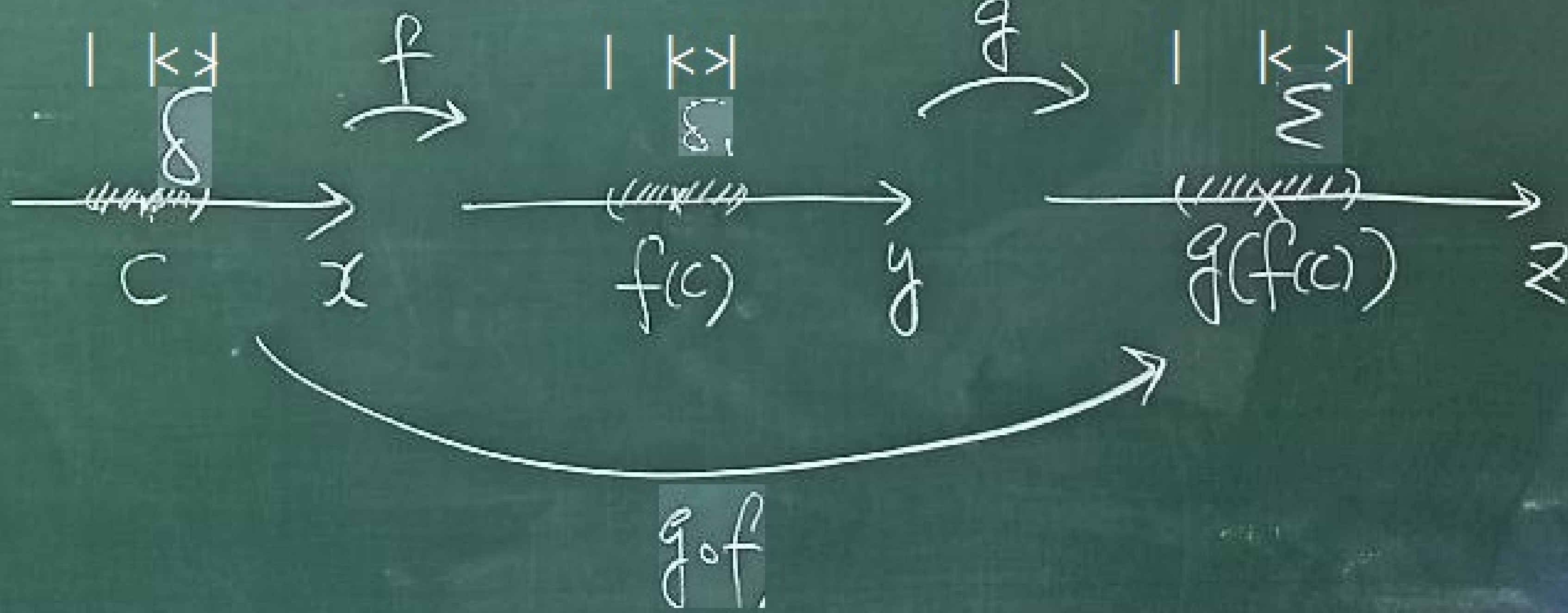
Thm 9 [Composite of continuous functions]

If f is continuous at c, g is cont. at $f(c)$. then $(g\circ f)(x)=g(f(x))$ is continuous at c.

i.e. $\lim\limits_{x\to c} g(f(x)) = g(f(c))$



Pf: To show that $g \circ f$ is continuous, we need to show for any $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that

"$|x - c| < \delta \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$"

To do this, we note that, for any $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that,

$(*_1)$ "$|y - f(c)| < \delta_1 \Rightarrow |g(y) - g(f(c))| < \varepsilon$"

(g is continuous at $f(c)$)

For this $\delta_1 > 0$, there exists a corresponding $\delta > 0$ such that

$(*_2)$ " $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1$ "

(f is continuous at c)

$(*_1) + (*_2) \Longrightarrow$

" $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1$

$\Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$ "

Eg1: $\lim_{x \to 0} \sqrt{x+1}\, 2^{\sin x} = \sqrt{0+1} \cdot 2^{\sin 0} = 1$

Since both $\sqrt{x+1}$ and $2^{\sin x}$ are continuous function (Thm 9), so is $\sqrt{x+1}\, 2^{\sin x}$ (Thm 8)

Remark

$\cos x$ is cont. at any $c \in \mathbb{R}$

Pf. $\lim_{x \to c} \cos x = \lim_{h \to 0} \cos(h+c)$ $(h = x-c)$

$= \lim_{h \to 0} (\cos h \cos c - \sin h \sin c)$

$= \cos c \lim_{h \to 0} (\cos h - 1 + 1) + \sin c \lim_{h \to 0} \sin h$

$= \cos c \left(\lim_{h \to 0} \frac{-2\sin^2 \frac{h}{2}}{h^2} \cdot \lim_{h \to 0} h^2 + 1 \right)$

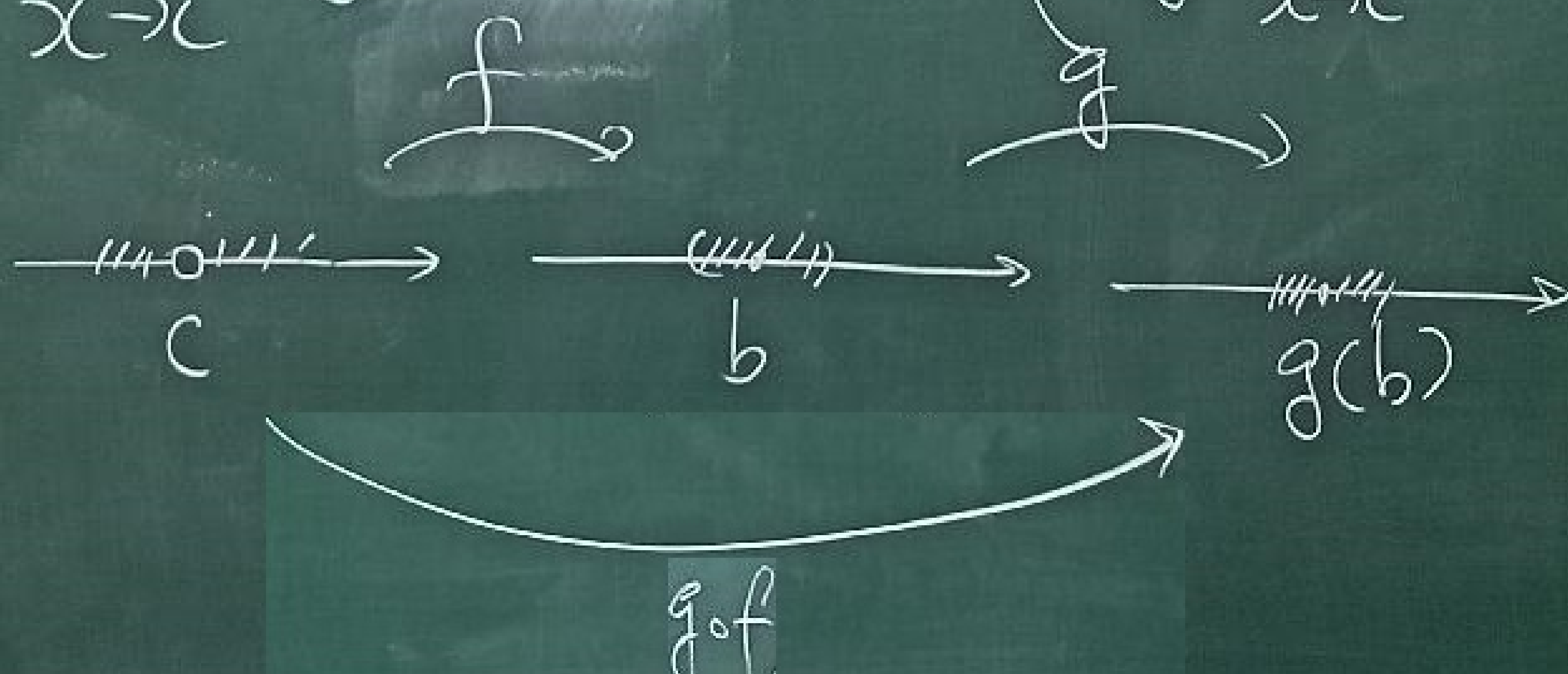
$+ \sin c \left(\lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} h \right)$

$= \cos c \cdot (0+1) + \sin c (1 \cdot 0)$

$= \cos c$

Thm 10 Similarly, if g is continuous at b and $\lim_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = g(b) \left(= g\left(\lim_{x\to c} f(x)\right)\right)$$



E.g $\displaystyle\lim_{\theta \to 0} \cos\left(\frac{\sin\theta}{\theta}\right) = ?$

<u>Ans</u> $\displaystyle \overset{\text{Thm 10}}{=} \cos\left(\lim_{\theta \to 0} \frac{\sin\theta}{\theta}\right)$

$= \cos 1$

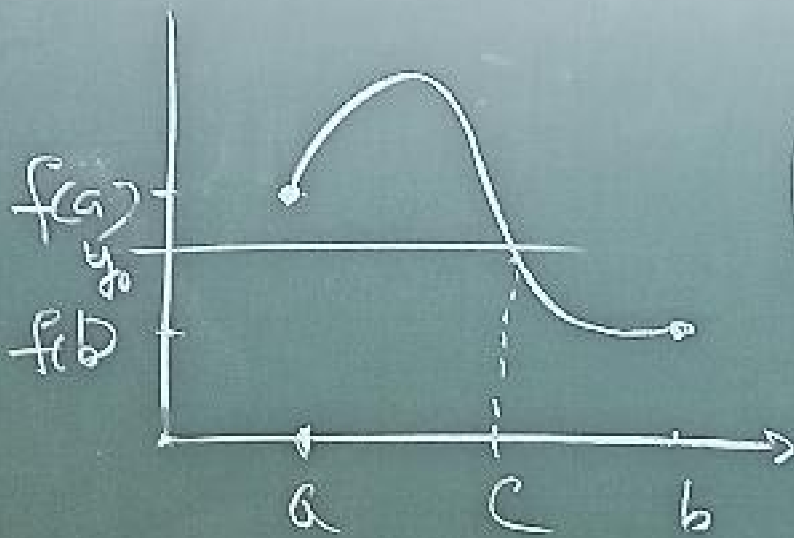
Thm 11 [Intermediate Value Thm]

f is continuous on $[a, b]$

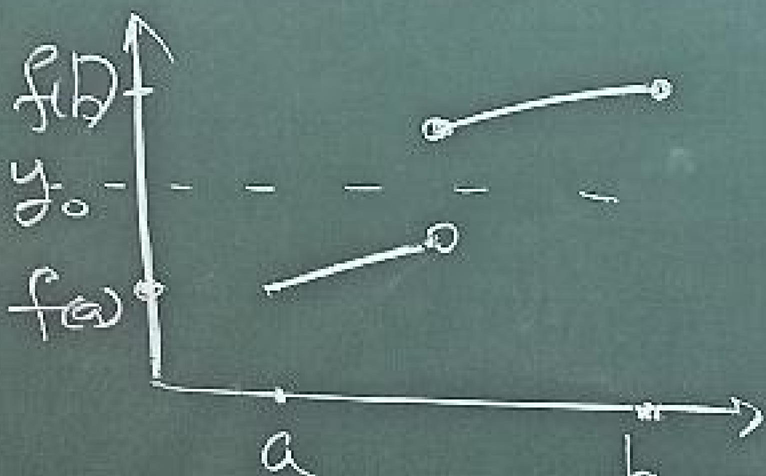
$\Longrightarrow$ f takes any value between $f(a)$ and $f(b)$.

i.e. if y_0 is any value between $f(a)$ and $f(b)$, then there exists a $c \in [a, b]$ such that

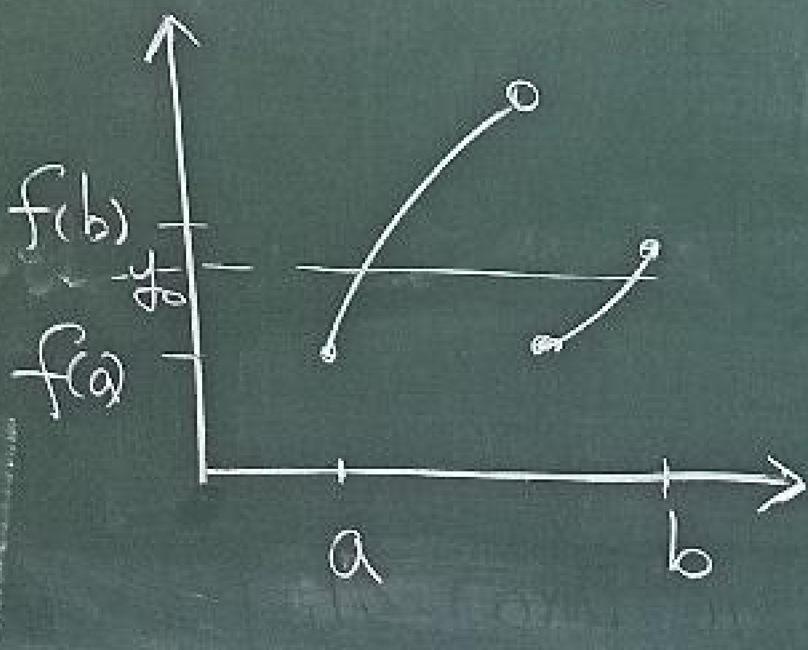
$$f(c) = y_0$$



(Typical case)



(I.V.T. may fail if f is not cont. on [a,b])

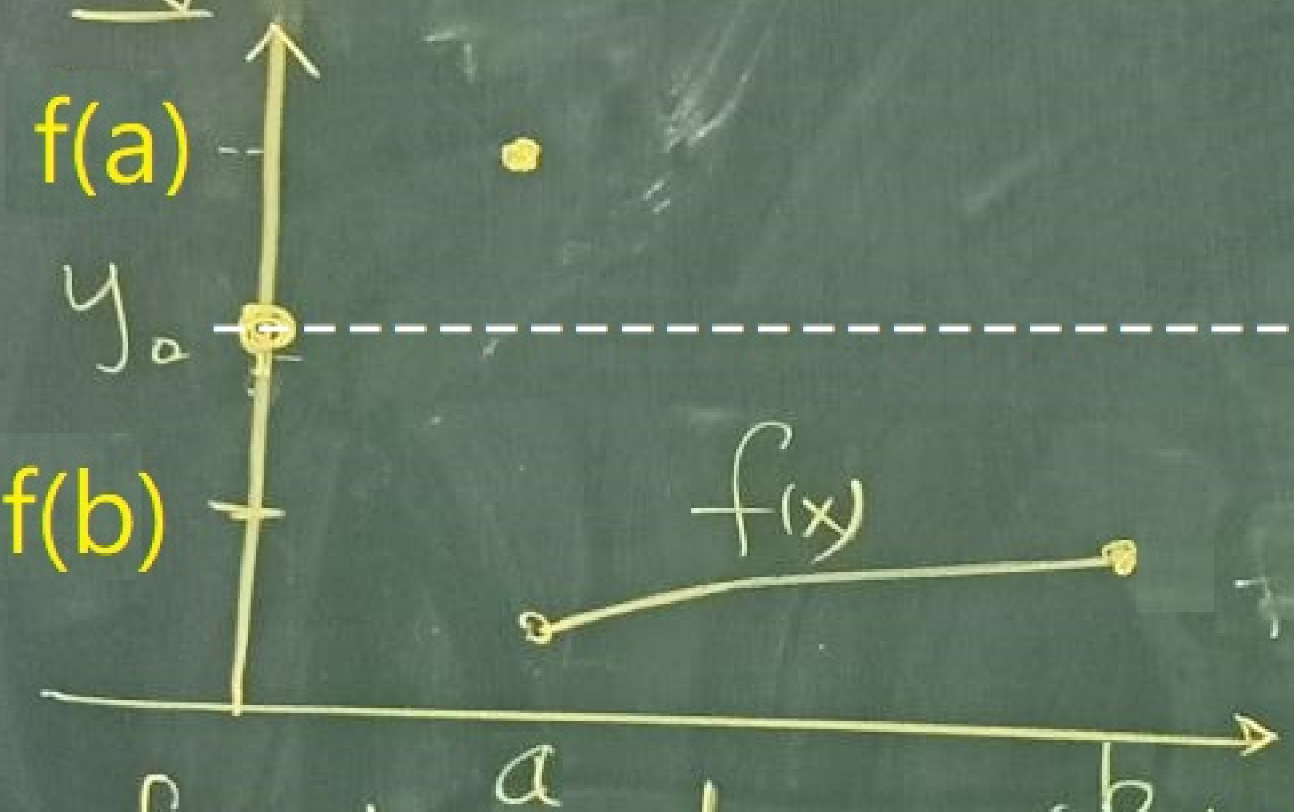


(f takes any value betwen f(a) and f(b) $\not\Rightarrow$ f is cont. on [a,b])

Remark

"f is cont. on $[a, b]$" is essential:

Eg



$f(x)$ is cont on $(a, b]$ but not on $[a, b]$

$\not\Rightarrow$ f takes any value between $f(a)$ and $f(b)$

Eg 2. Show that $f(x) = x^3 - x - 1 = 0$ has a root in $(1, 2)$

Sol. $f(1) = -1$, $f(2) = 5$.

$\underset{\Longrightarrow}{\text{I.V.T.}}$ $\exists\, x \in (1, 2)$, $f(x) = 0$.
(存在)

Eg 3. Show that

$\sqrt{2x+5} = 4 - x^2$ has a solution.

Sol. Let $f(x) = \sqrt{2x+5} - (4 - x^2)$

$f(0) = \sqrt{5} - 4 < 0$, $f(2) = 3 - 0 > 0$

I.V.T. $\Rightarrow \exists\, x \in (0, 2)$, $f(x) = 0$, $\left(\sqrt{2x+5} = 4 - x^2\right)$