One sided limit

Suppose $f(x)$ is defined on $(c, c+a)$, $a>0$

$\left(\underline{(c-a, c)} \right)$

Def: $\lim\limits_{x \to c^{\underline{\pm}}} f(x) = L$

if for any $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that

" $c < x < c + \delta \Longrightarrow |f(x) - L| < \varepsilon$ "

$\left(\underline{c - \delta < x < c} \right)$

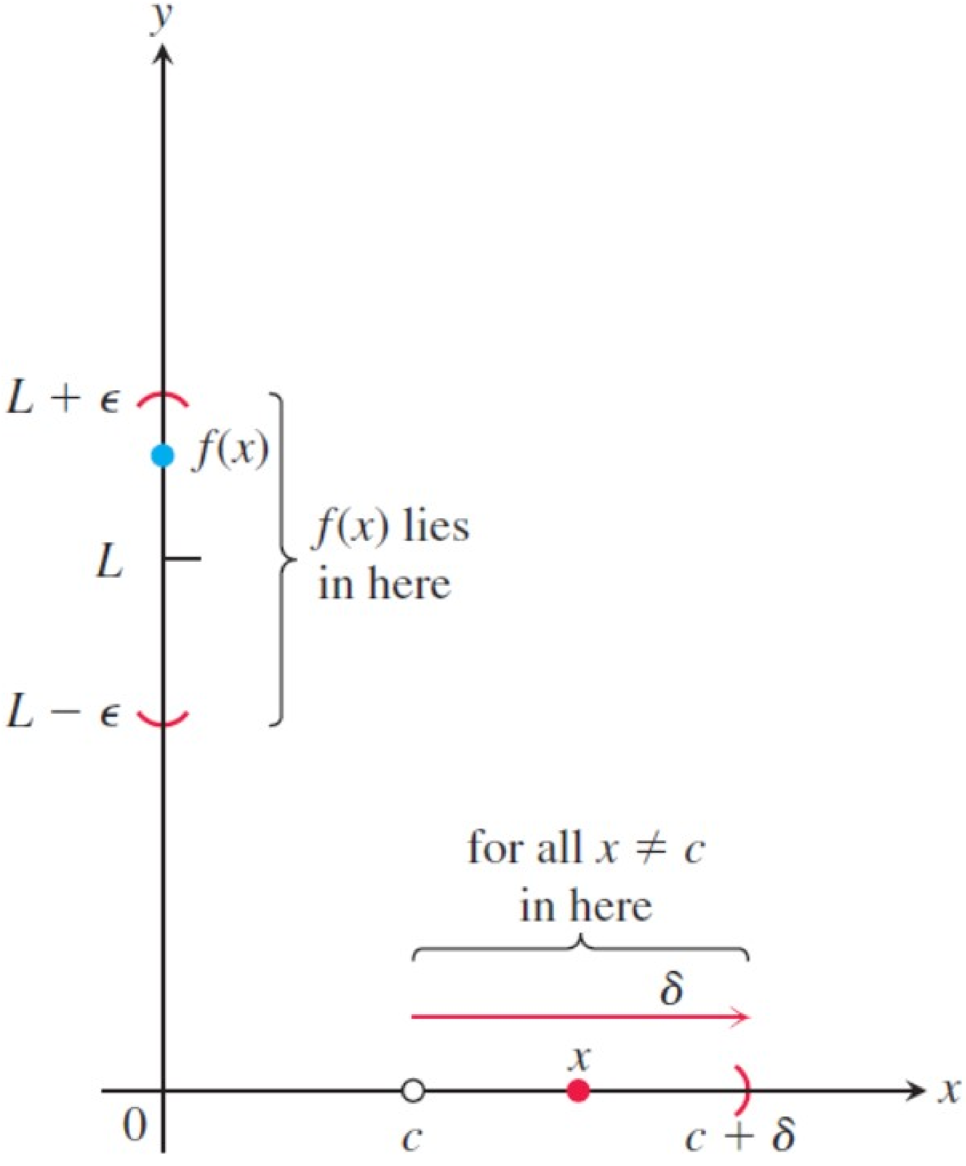


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as do the Sandwich Theorem and Theorem 5. One-sided limits are related to limits in the following way.

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Eg4. Show that $\lim_{x\to 0^+} \sin\frac{1}{x}$ does not exist.

Pf Let $a_n = \frac{1}{n\pi}$

$b_n = \frac{1}{(2n+\frac{1}{2})\pi}$, $c_n = \frac{1}{(2n-\frac{1}{2})\pi}$

$n = 1, 2, 3, \cdots$

$\Rightarrow \sin\frac{1}{a_n} = 0$, $\sin\frac{1}{b_n} = 1$, $\sin\frac{1}{c_n} = -1$

for any $\delta > 0$, there are infinitly infinitely many a_n, b_n and c_n on $(0, \delta)$

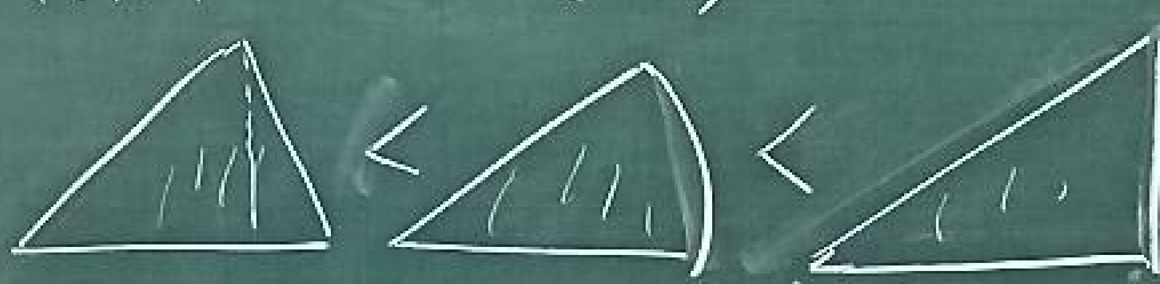
$\Rightarrow 0 < \begin{matrix} a_n \\ b_n \\ c_n \end{matrix} < \delta \Rightarrow \begin{matrix} f(a_n) = 0 \\ f(b_n) = 1 \\ f(c_n) = -1 \end{matrix} \not\Rightarrow \left| f\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} - L \right| < \frac{1}{2}$ (any $L \in \mathbb{R}$)

$\therefore$ when $\varepsilon = \frac{1}{2}$, corresponding δ does not exist the lim does not exist

Limits involving $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ $(=?)$

Sol: We start with $\lim_{\theta \to 0^+}$

$\left(0<\theta<\frac{\pi}{2}\right)$



$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}$$

$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim_{\theta \to 0^+} \cos\theta = 1 = \lim_{\theta \to 0^+} 1$

From Sandwich theorem
(one sided version)

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = \underline{1}$$

What if $\frac{-\pi}{2} < \theta < 0$?

Let $\theta = -\varphi$. $0 < \varphi < \frac{\pi}{2}$

$$\Rightarrow 1 > \frac{\sin\varphi}{\varphi} > \cos\varphi$$

$\sin\varphi = -\sin\theta$, $\cos\varphi = \cos\theta$

$\Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$ also on $\frac{-\pi}{2} < \theta < 0$

From Sandwich Thm $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

(since lim cos(theta)=1
and lim 1 = 1)

Continuity

Suppose $f(x)$ is defined on $[a,b]$

If $c \in (a,b)$, then f is

$(c \in (a,b],\ c \in [a,b))$

continuous at c, if

(left continuous, right continuous)

$$\lim_{x \to c} f(x) = f(c)$$

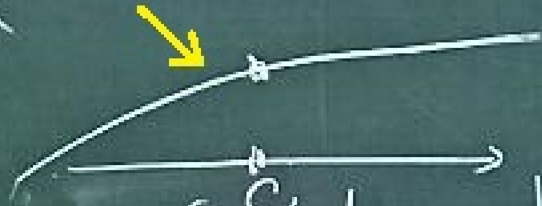
$(x \to c^-,\ x \to c^+)$

That is, for any $\varepsilon>0$,
there exists a corresponding $\delta>0$
such that

"$|x-c|<\delta \Rightarrow |f(x)-f(c)|<\varepsilon$"

$(c-\delta<x\le c,\ c\le x<c+\delta)$

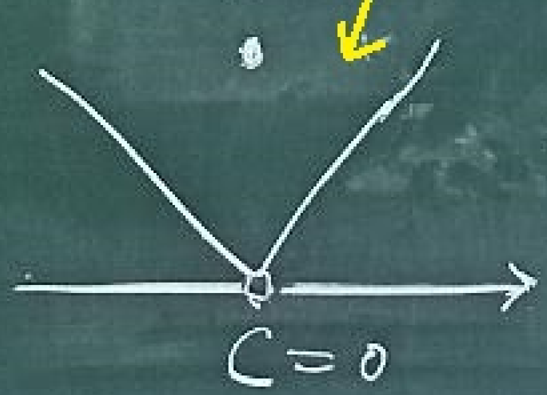
(continuous at c)

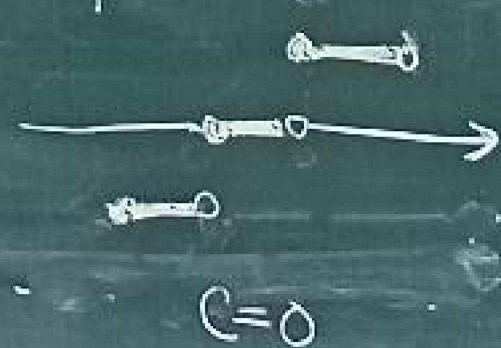


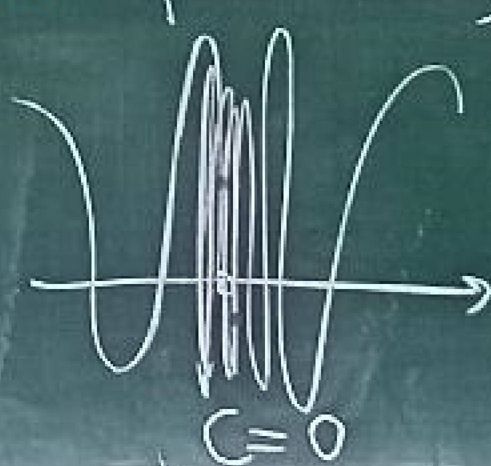
(Not continuos at c)

$f(x)=\lfloor x\rfloor$

$f(x)=\sin\frac{1}{x}$







Remark: f is a continuous function if f is continuous at every point of its domain.

Basic properties of continuous functions (See Section 2.5, Thm 8)

Remark: polynomials, rational functions, trigonometric functions are all continuous at the interior of their domain of definition,

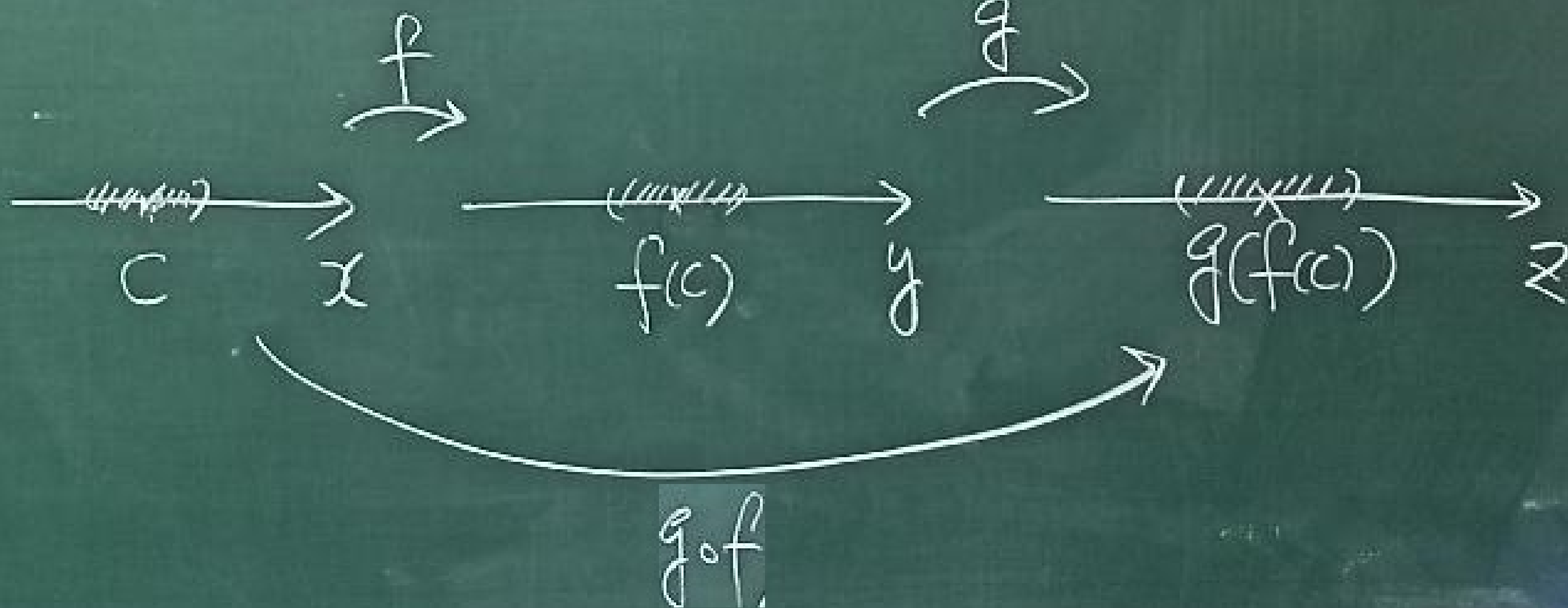
THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. *Sums:*	$f + g$	
2. *Differences:*	$f - g$	
3. *Constant multiples:*	$k \cdot f$,	for any number k
4. *Products:*	$f \cdot g$	
5. *Quotients:*	f/g,	provided $g(c) \neq 0$
6. *Powers:*	f^n,	n a positive integer
7. *Roots:*	$\sqrt[n]{f}$,	provided it is defined on an open interval containing c, where n is a positive integer

Thm 9 [Composite of continuous functions]

If f is continuous at c, g is cont. at $f(c)$. then $(g\circ f)(x) = g(f(x))$ is continuous at c.

i.e. $\lim\limits_{x\to c} g(f(x)) = g(f(c))$



Pf: To show that $g \circ f$ is continuous, we need to show for any $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that
"$|x-c| < \delta \Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$"
To do this, we note that, for any $\varepsilon > 0$ there exists a $\delta_1 > 0$ such that,
$(*_1)$ "$|y - f(c)| < \delta_1 \Rightarrow |g(y) - g(f(c))| < \varepsilon$"
(g is continuous at $f(c)$)

For this $\delta_1 > 0$, there exists a corresponding $\delta > 0$ such that

"$(*_2)$ $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1$"

(f is continuous at c)

$(*_1) + (*_2) \Longrightarrow$

"$|x-c| < \delta \Rightarrow |f(x) - f(c)| < \delta_1$
$\Rightarrow |g(f(x)) - g(f(c))| < \varepsilon$"

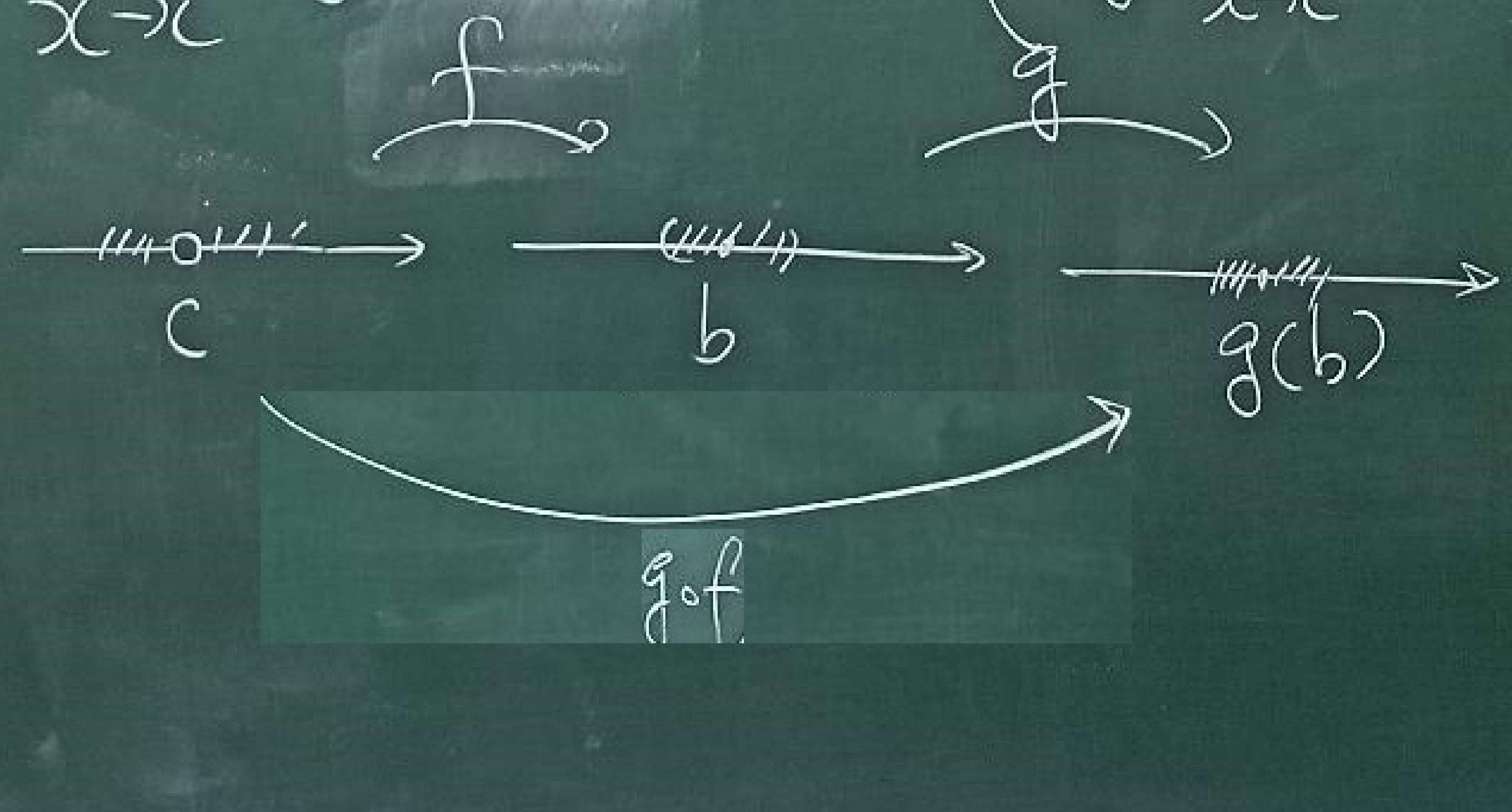
Eg1: $\lim_{x \to 0} \sqrt{x+1}\, 2^{\sin x} = \sqrt{0+1} \cdot 2^{\sin 0} = 1$

Since both $\sqrt{x+1}$ and $2^{\sin x}$ are continuous function (Thm 9), so is $\sqrt{x+1}\, 2^{\sin x}$ (Thm 8)

Thm 10 Similarly, if g is continuous at b and

$$\lim_{x\to c} f(x) = b, \text{ then}$$

$$\lim_{x\to c} g(f(x)) = g(b) \left(= g\left(\lim_{x\to c} f(x)\right)\right)$$



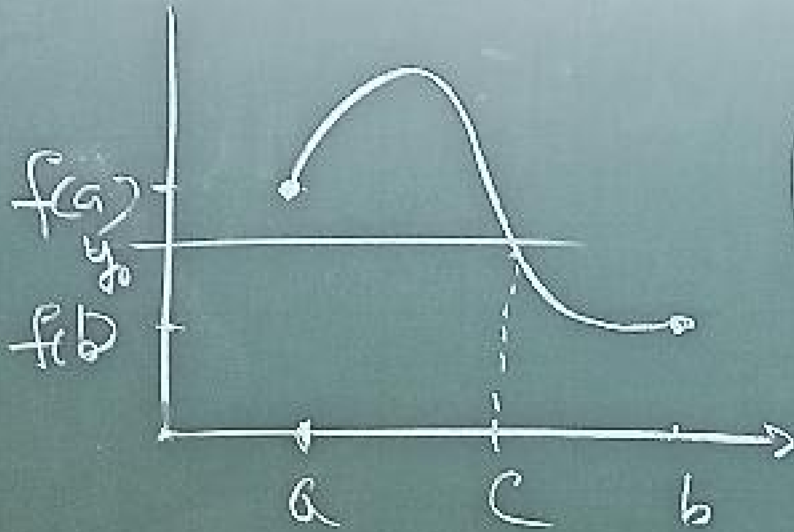
Thm 11 [Intermediate Value Thm]

f is continuous on $[a, b]$

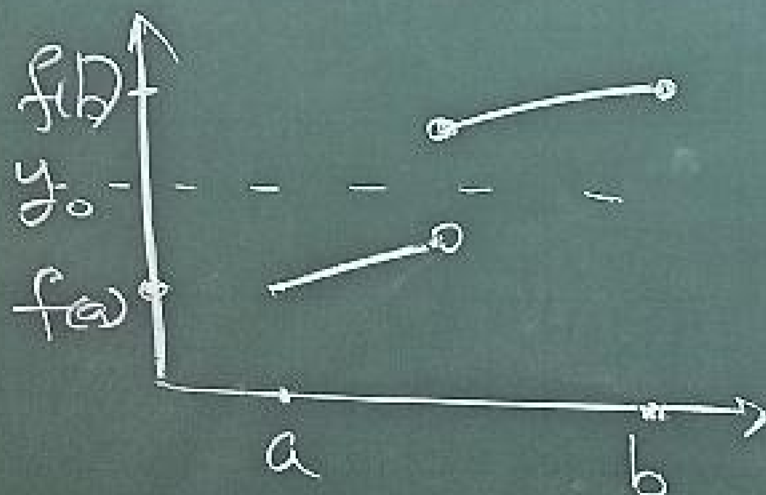
$\Longrightarrow$ f takes any value between $f(a)$ and $f(b)$.

i.e. if y_0 is any value between $f(a)$ and $f(b)$, then there exists a $c \in [a, b]$ such that

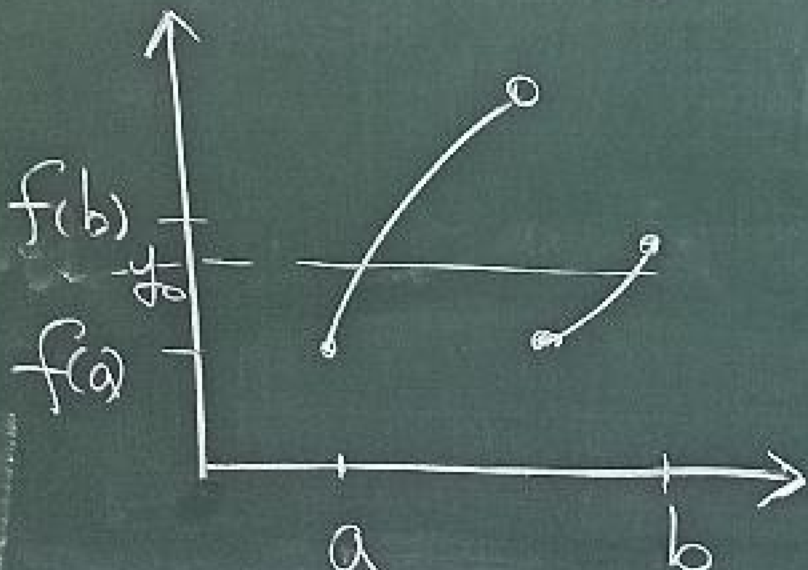
$$f(c) = y_0$$



(Typical case)



(I.V.T. may fail if f is not cont. on [a,b])



(f takes any value betwen f(a) and f(b) $\not\Rightarrow$ f is cont. on [a,b])

Eg 2. Show that $f(x)=x^3-x-1=0$ has a root in $(1,2)$

Sol. $f(1)=-1$, $f(2)=5$.

$\overset{\text{I.V.T.}}{\Longrightarrow} \underset{(存在)}{\exists} x\in(1,2).\ f(x)=0$.

Eg 3. Show that

$\sqrt{2x+5}=4-x^2$ has a solution.

Sol. Let $f(x)=\sqrt{2x+5}-(4-x^2)$

$f(0)=\sqrt{5}-4<0$, $f(2)=3-0>0$

$\text{I.V.T.} \Rightarrow \exists x\in(0,2),\ f(x)=0,\ (\sqrt{2x+5}=4-x^2)$