One sided limit

Suppose $f(x)$ is defined

on $(c, c+a)$, $a > 0$

$((c-a, c))$

Def: $\lim_{x \to c^{\pm}} f(x) = L$

if for any $\varepsilon > 0$, there exists a corresponding $\delta > 0$, such that

" $c < x < c+\delta \Longrightarrow |f(x) - L| < \varepsilon$ "

$(c-\delta < x < c)$

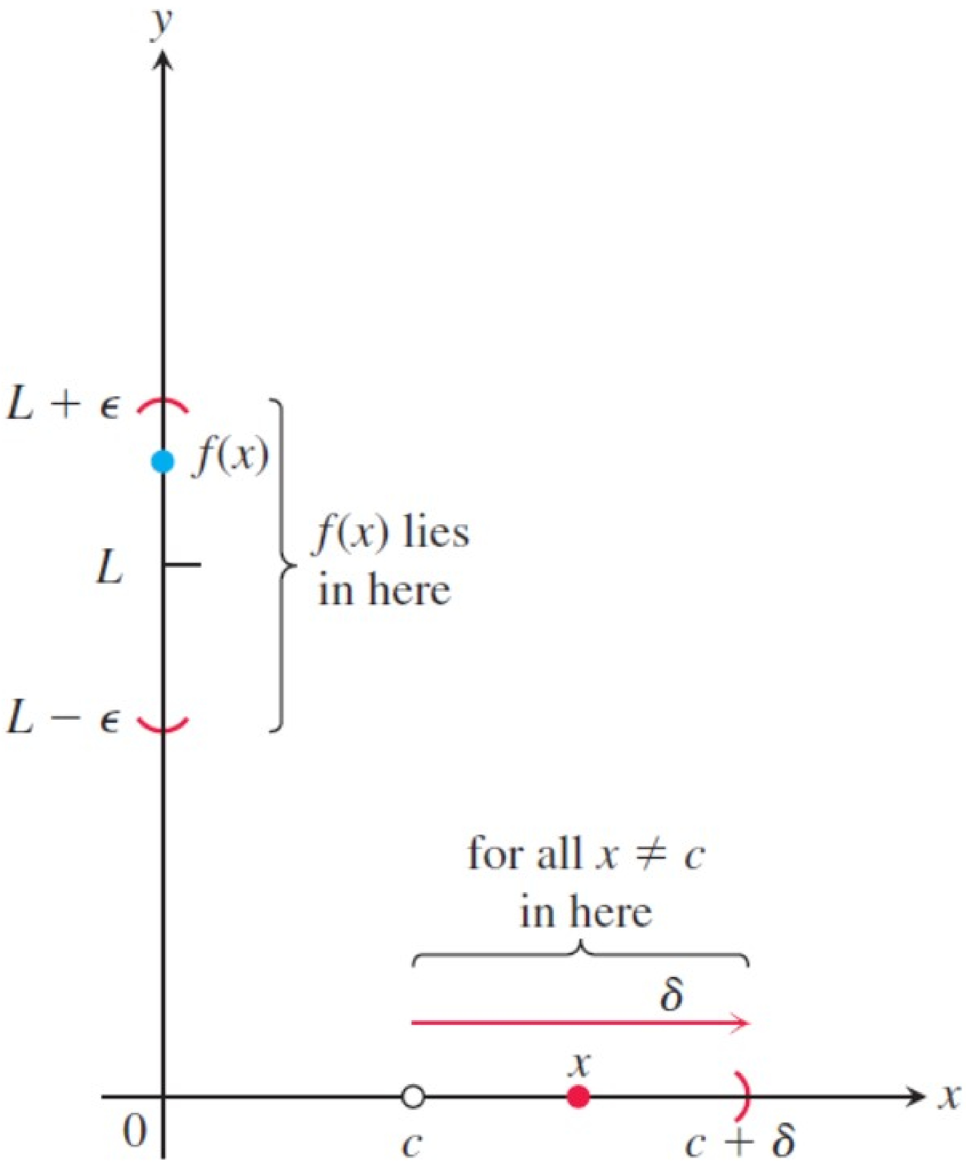


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as do the Sandwich Theorem and Theorem 5. One-sided limits are related to limits in the following way.

THEOREM 6 A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

Eg 3. Show that $\lim_{x \to 0^+} \sqrt{x} = 0$

Pf For any $\varepsilon > 0$, find $\delta > 0$ such that

$$\text{"}0 < x < \delta \Longrightarrow |\sqrt{x} - 0| < \varepsilon\text{"}$$

$$|\sqrt{x} - 0| < \varepsilon$$

$$\Leftrightarrow \sqrt{x} - 0 < \varepsilon$$

$$\Leftrightarrow x < \varepsilon^2, \ (x > 0)$$

$$\Leftrightarrow 0 < x < \varepsilon^2$$

We only need the "<=" part

Take $\delta = \varepsilon^2$ #

Remark:

"$\lim_{x \to c^+} f(x) = L$ is false"

There exists an $\varepsilon > 0$ (find this ε!)

such that for any $\delta > 0$

"$0 < x - c < \delta \not\Longrightarrow |f(x) - L| < \varepsilon$"

(i.e. $\exists\, x^*$, $0 < x^* - c < \delta$, "$|f(x^*) - L| \geq \varepsilon$")

Eg 4. Show that $\lim\limits_{x\to 0^+} \sin\frac{1}{x}$ does not exist.

Pf Let $a_n=\frac{1}{n\pi}$

$b_n=\frac{1}{(2n+\frac{1}{2})\pi}$, $c_n=\frac{1}{(2n-\frac{1}{2})\pi}$

$n=1,2,3,\cdots$

$\Rightarrow \sin\frac{1}{a_n}=0$, $\sin\frac{1}{b_n}=1$, $\sin\frac{1}{c_n}=-1$

for any $\delta>0$, there are infintly infinitely many a_n, b_n and c_n on $(0,\delta)$

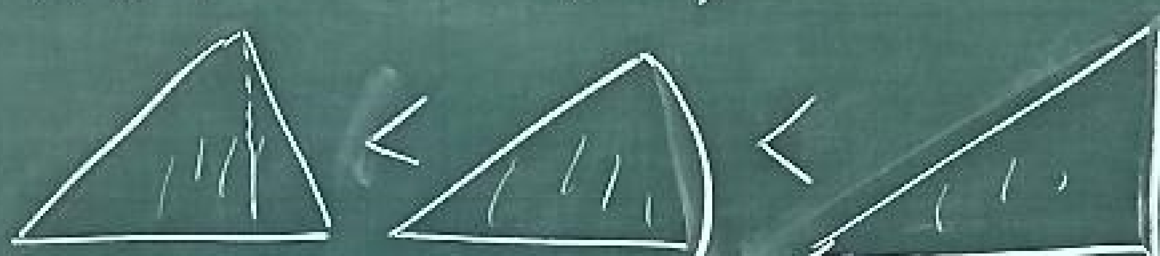
$$\Rightarrow 0<\begin{matrix}a_n\\b_n\\c_n\end{matrix}<\delta \Rightarrow \begin{matrix}f(a_n)=0\\f(b_n)=1\\f(c_n)=-1\end{matrix} \not\Rightarrow \left|f\begin{pmatrix}a_n\\b_n\\c_n\end{pmatrix}-L\right|<\frac{1}{2} \quad (\text{any } L\in\mathbb{R})$$

$\therefore$ when $\varepsilon=\frac{1}{2}$, corresponding δ does not exist the lim does not exist

Limits involving $\lim_{\theta\to 0} \frac{\sin\theta}{\theta}$ (=?)

Sol: We start with $\lim_{\theta\to 0^+}$

$\left(0<\theta<\frac{\pi}{2}\right)$



$$\frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}$$

$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim_{\theta \to 0^+} \cos\theta = 1 = \lim_{\theta \to 0^+} 1$

From Sandwich theorem (one sided version)

$$\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$$

What if $\frac{-\pi}{2} < \theta < 0$?

Let $\theta = -\varphi$. $0 < \varphi < \frac{\pi}{2}$

$$\Rightarrow 1 > \frac{\sin\varphi}{\varphi} > \cos\varphi$$

$\sin\varphi = -\sin\theta$, $\cos\varphi = \cos\theta$

$\Rightarrow 1 > \frac{\sin\theta}{\theta} > \cos\theta$ also on $\frac{-\pi}{2} < \theta < 0$

From Sandwich Thm $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

(since lim cos(theta)=1 and lim 1 = 1)

Eg 5(a) $\lim_{h\to 0} \frac{\cos h - 1}{h} = ?$

Sol

$$= \lim_{h\to 0} \frac{-2\sin^2\frac{h}{2}}{h}$$

$$= \lim_{h\to 0} -\cancel{2}\,\frac{\sin\frac{h}{2}}{\cancel{2}\,\frac{h}{2}}\,\sin\frac{h}{2}$$

$$= -1 \cdot \lim_{h\to 0} \sin\left(\frac{h}{2}\right)$$

$$= -1 \cdot 0 = 0$$

5(b) $\lim_{x\to 0} \frac{\sin 2x}{5x} = ?$

Sol

$$= \lim_{x\to 0} \frac{\sin 2x}{2\cancel{x}} \cdot \frac{2\cancel{x}}{5\cancel{x}}$$

$$= 1 \cdot \frac{2}{5}$$

Eg 6. $\lim_{t \to 0} \dfrac{\tan t \sec 2t}{3t} = ?$

Sol

$$= \lim_{t \to 0} \frac{\frac{\sin t}{\cos t} \cdot \frac{1}{\cos 2t}}{3t}$$

$$= \lim_{t \to 0} \frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{3} \cdot \frac{1}{\cos 2t}$$

$$= \lim_{t \to 0} 1 \cdot 1 \cdot \frac{1}{3} \cdot 1 = \frac{1}{3}$$

Continuity

Suppose $f(x)$ is defined on $[a,b]$

If $c \in (a,b)$, then f is

($c \in (a,b]$, $c \in [a,b)$)

continuous at c, if

(left continuous, right continuous)

$$\lim_{x \to c} f(x) = f(c)$$

($x \to c^-$, $x \to c^+$)

That is, for any $\varepsilon > 0$,

there exists a corresponding $\delta > 0$

such that

"$|x-c| < \delta \Rightarrow |f(x)-f(c)| < \varepsilon$"

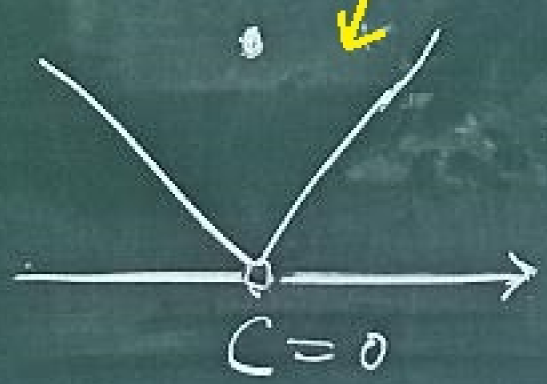
$(c-\delta < x \leq c, \quad c \leq x < c+\delta)$

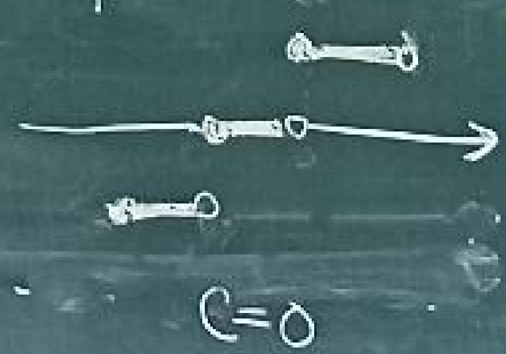
(continuous at c)

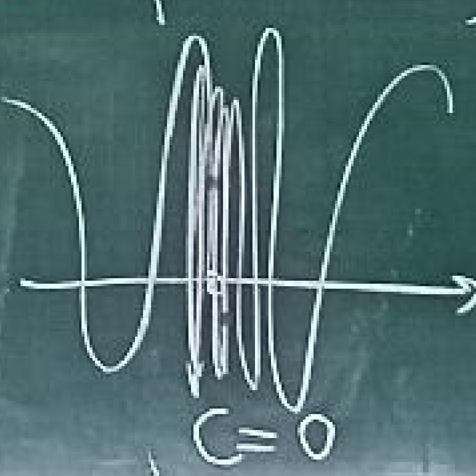
(Not continuos at c)

$f(x) = \lfloor x \rfloor$

$f(x) = \sin\frac{1}{x}$







<u>Remark</u>: f is a continuous function if f is continuous at every point of its domain.

Basic properties of continuous functions (See Section 2.5, Thm 8)

<u>Remark</u>: polynomials, rational functions, trigonometric functions are all continuous at the interior of their domain of definition,

THEOREM 8—Properties of Continuous Functions If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1.	*Sums:*	$f + g$
2.	*Differences:*	$f - g$
3.	*Constant multiples:*	$k \cdot f$, for any number k
4.	*Products:*	$f \cdot g$
5.	*Quotients:*	f/g, provided $g(c) \neq 0$
6.	*Powers:*	f^n, n a positive integer
7.	*Roots:*	$\sqrt[n]{f}$, provided it is defined on an open interval containing c, where n is a positive integer

Thm 9 [Composite of continuous functions]

If f is continuous at c, g is cont. at $f(c)$. then $(g \circ f)(x) = g(f(x))$ is continuous at c.

i.e. $\lim_{x \to c} g(f(x)) = g(f(c))$