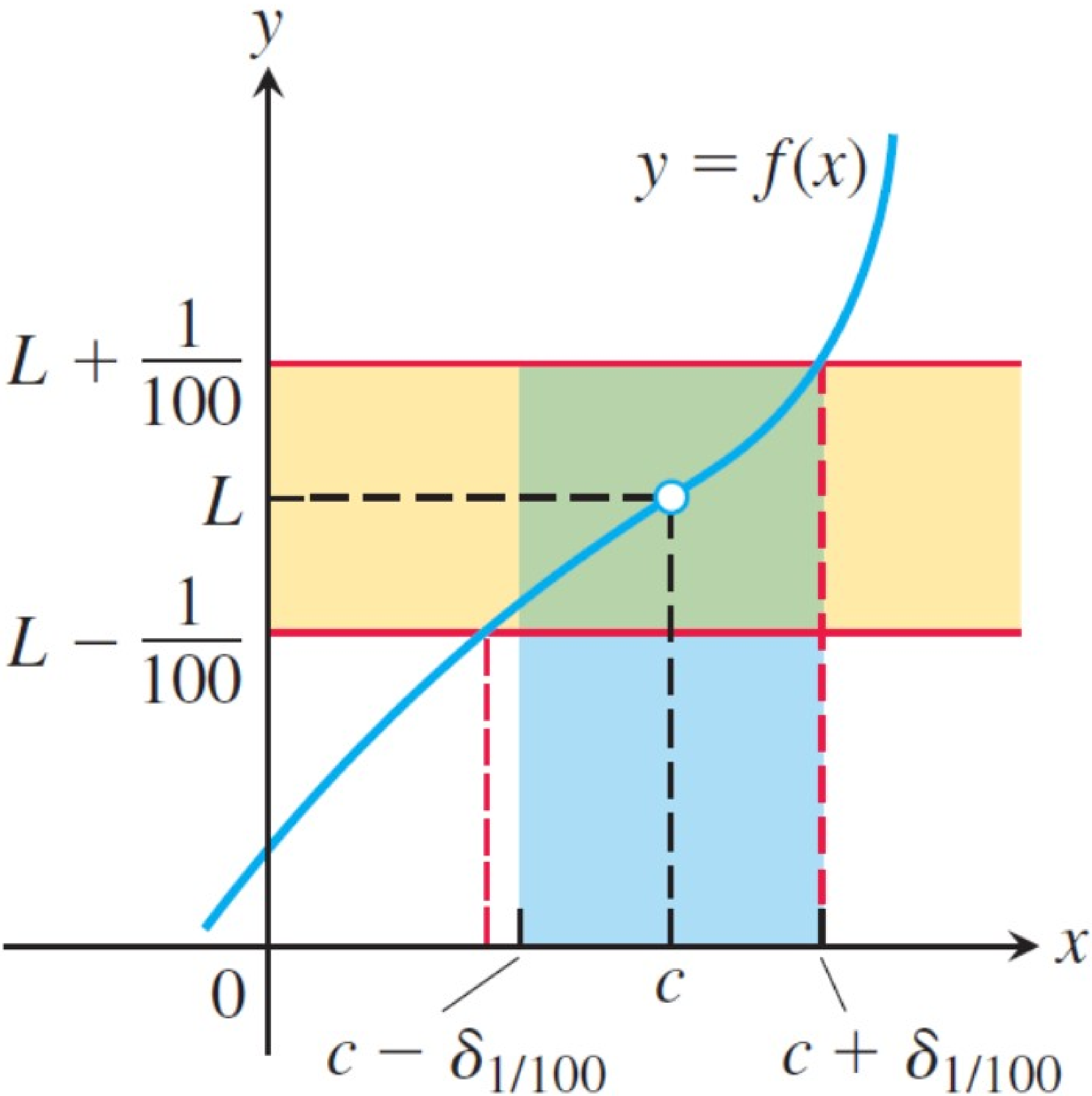
~~Suppose $f(x)$ is defined on $(c-\delta, c)\cup(c, c+\delta)$, $\delta>0$~~

Def: $\lim\limits_{x\to c} f(x) = L$

if for every $\varepsilon>0$, there exists
(對每個正數ε，皆存在)
a corresponding $\delta>0$, such that
(一個對應的正數δ，使得下式成立)

"$0<|x-c|<\delta \implies |f(x)-L|<\varepsilon$" (1)

("若(x滿足) $0<|x-c|<\delta$，則 $|f(x)-L|<\varepsilon$")



Response:

$$|x - c| < \delta_{1/100}$$

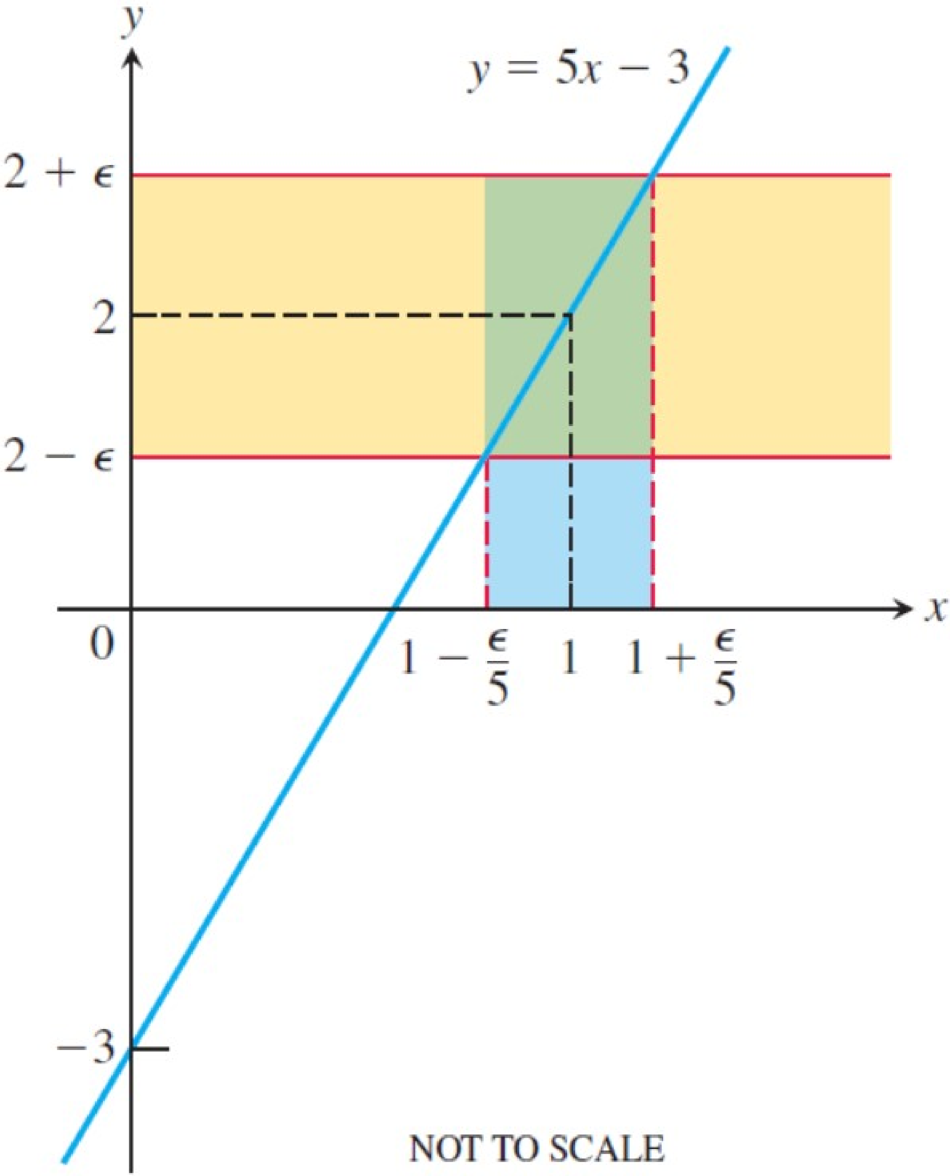


FIGURE 2.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \epsilon/5$ guarantees that $|f(x) - 2| < \epsilon$ (Example 2).

How to prove $\lim\limits_{x\to c} f(x) = L$?

Ans: For any $\varepsilon > 0$, describe how to find corresponding $\delta > 0$

Eg [redacted] Use definition to show that

$$\lim_{x \to 0} x^3 = 0$$

Pf: For $\varepsilon > 0$, Want to find $\delta > 0$ such that

"$0 < |x - 0| < \delta \Rightarrow |x^3 - 0| < \varepsilon$"

$$|x^3 - 0| < \varepsilon$$

$$\Leftrightarrow -\varepsilon < x^3 - 0 < \varepsilon$$

$$\Leftrightarrow -\varepsilon^{\frac{1}{3}} < x < \varepsilon^{\frac{1}{3}}$$

$$\Leftrightarrow |x| < \varepsilon^{\frac{1}{3}}$$

$$\Leftarrow 0 < |x - 0| < \varepsilon^{\frac{1}{3}}$$

Take $\delta = \varepsilon^{\frac{1}{3}}$, then

$$0 < |x - 0| < \delta \Rightarrow |x^3 - 0| < \varepsilon$$

※

Eg 4: Show that $\lim_{x \to 5} \sqrt{x-1} = 2$

Pf: For every $\varepsilon > 0$ ($0 < \varepsilon < 4$)

$$|\sqrt{x-1} - 2| < \varepsilon$$

$$\Leftrightarrow -\varepsilon < \sqrt{x-1} - 2 < \varepsilon$$

$$\Leftrightarrow (2-\varepsilon)^2 < x-1 < (2+\varepsilon)^2$$

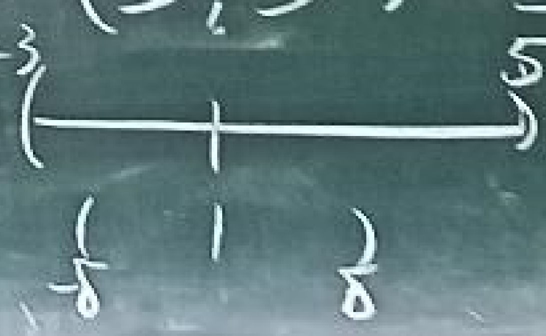
$$\Leftrightarrow (2-\varepsilon)^2 - 4 < x-5 < (2+\varepsilon)^2 - 4$$

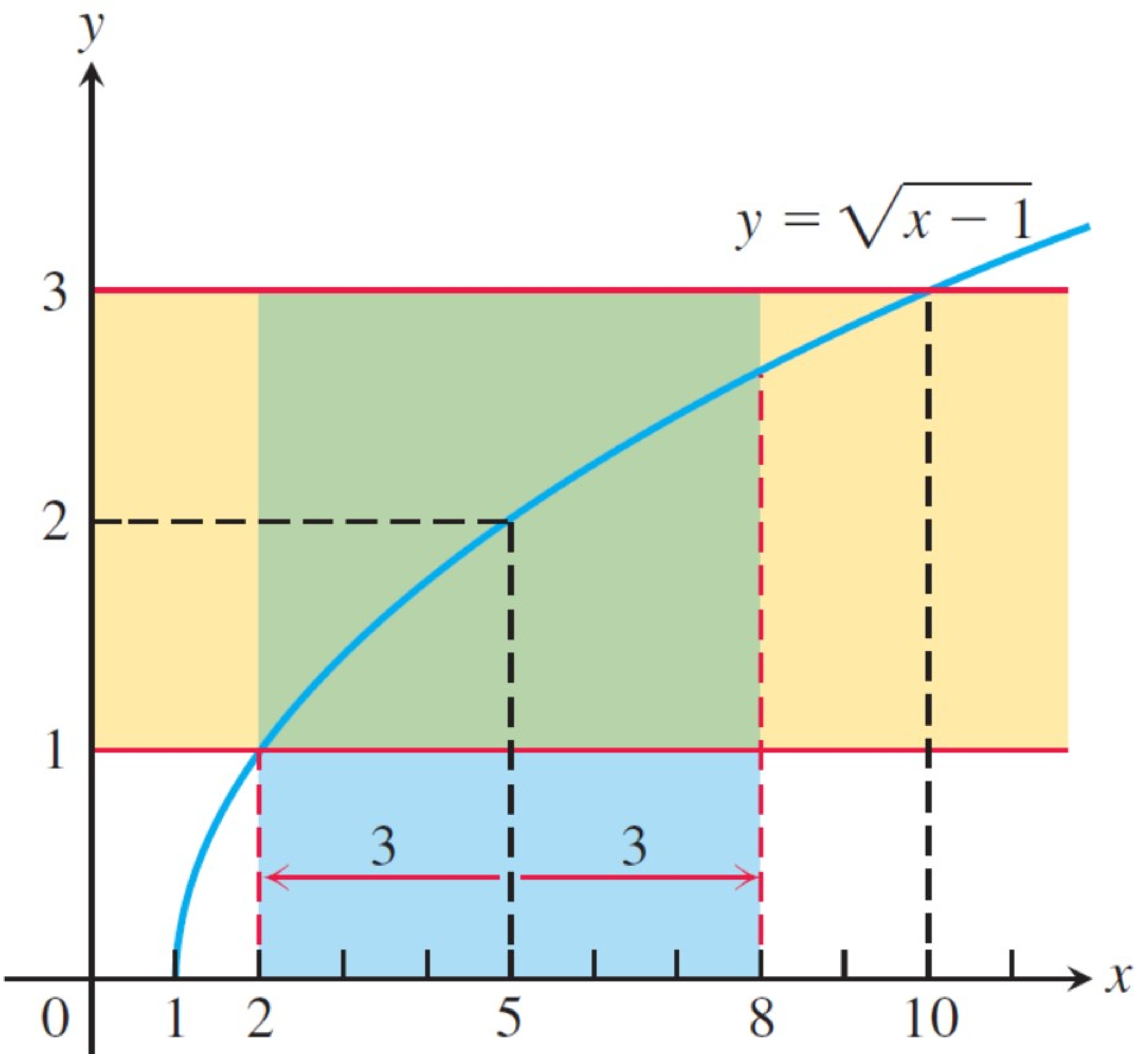
(We demostrate by $\varepsilon = 1$)

$$-3 < x-5 < 5$$

take $\delta = \min(3, 5) = 3$

$$\Leftarrow 0 < |x-5| < \delta$$





NOT TO SCALE

The method of finding delta for (epsilon=1) on previous page also works for general $0 < \text{epsilon} < 4$

For general $0<\varepsilon<4$

$$(2-\varepsilon)^2-4 < x-5 < (2+\varepsilon)^2-4$$

"$-$" if $0<\varepsilon<4$ "$+$"

Take $\delta = \min\left(\underset{\text{"}+\text{"}}{4-(2-\varepsilon)^2}, \underset{\text{"}+\text{"}}{(2+\varepsilon)^2-4}\right) > 0$

$\Leftarrow \quad 0<|x-5|<\delta$

Remark:

In general, we only need to find $\delta>0$ for every ε in $0<\varepsilon<\varepsilon_0$

$\because$ If $\varepsilon_1<\varepsilon_2$ and "$0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon_1$"

$\Rightarrow$ "$0<|x-c|<\delta \Rightarrow |f(x)-L|<\varepsilon_2$"

What if $\varepsilon \geq 4$?

Take $\varepsilon = 3$, the corresponding $\delta = \delta(3)$

then

$$0 < |x-5| < \delta(3)$$

$$\Rightarrow |f(x)-2| < 3 < 4 \leq \varepsilon$$

Eg 5 $f(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$

Prove that $\lim\limits_{x \to 2} = 4$

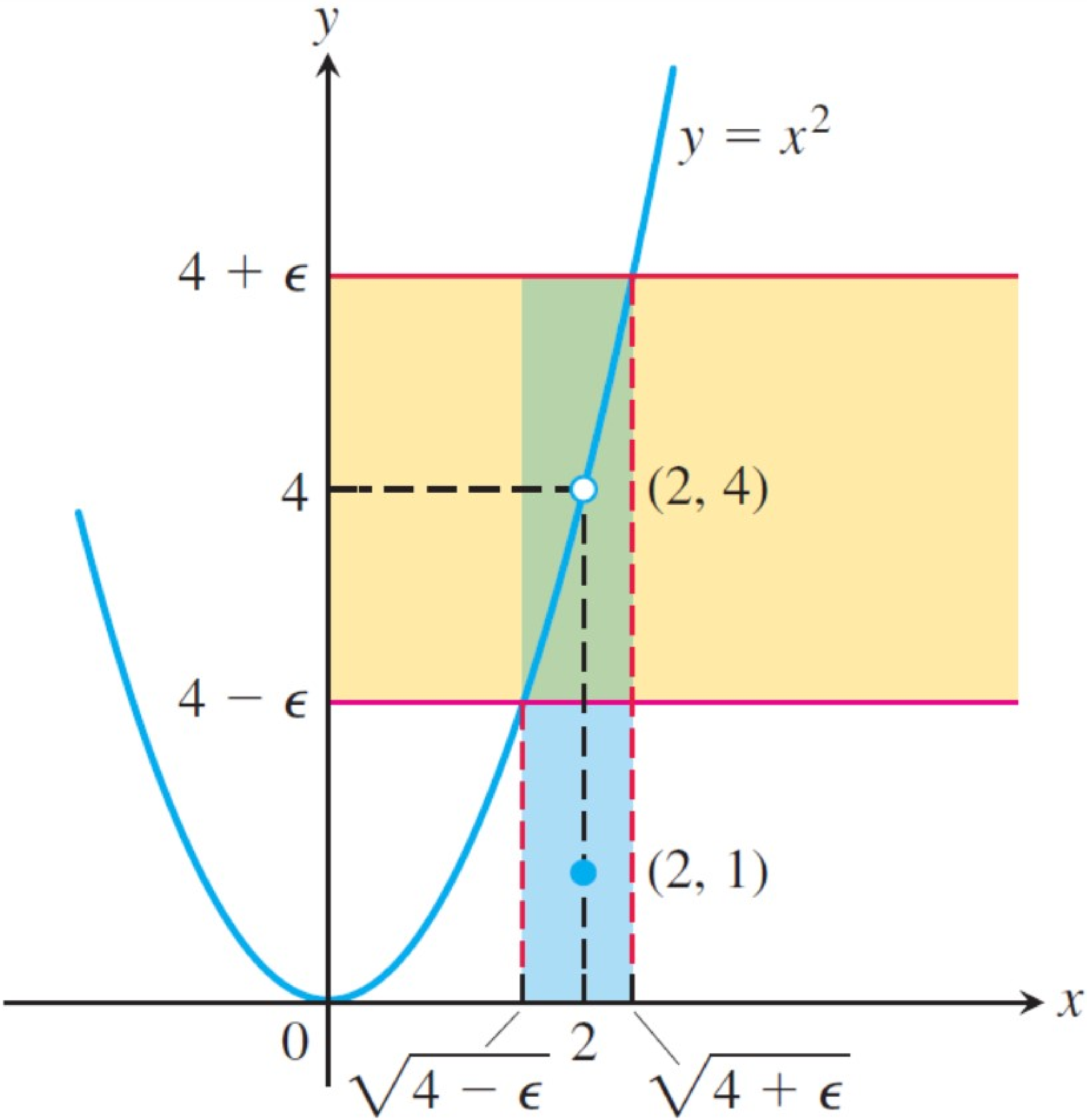
pf For any $\varepsilon > 0$, $(0 < \varepsilon \leq 4)$

we need ~~Need~~ to find $\delta > 0$

such that

$$|x^2 - 4| < \varepsilon, \quad \left(\begin{array}{c}\text{when} \\ x \neq 2\end{array}\right)$$

$$\Leftarrow 0 < |x - 2| < \delta$$



y
$y = x^2$
$4 + \epsilon$
4
(2, 4)
$4 - \epsilon$
(2, 1)
x
0
2
$\sqrt{4 - \epsilon}$
$\sqrt{4 + \epsilon}$

Only the "<=" part of "<=>" is relevant.

$$|x^2-4|<\varepsilon,\ (x\neq 2)$$

$$\Leftrightarrow -\varepsilon < x^2-4 < \varepsilon,\ (x \neq 2)$$

$$\Leftrightarrow 4-\varepsilon < x^2 < 4+\varepsilon,\ (x\neq 2)$$

$$\Leftrightarrow \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon},\ (x\neq 2)$$

(Need $4-\varepsilon \geq 0$)

(or $\underline{-\sqrt{4+\varepsilon} < x < -\sqrt{4-\varepsilon}}$

Not the interval we want)

$$\Leftarrow \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon},\ x\neq 2$$

$$\Leftrightarrow \underbrace{\sqrt{4-\varepsilon}-2}_{\text{"}-\text{"}} < x-2 < \underbrace{\sqrt{4+\varepsilon}-2}_{\text{"}+\text{"}}\ (x\neq 2)$$

Find $\delta>0$ such that

$\Leftarrow 0<|x-2|<\delta$

We need

(i) $\sqrt{4-\varepsilon}-2<0$ (satisfied if $\because 0<\varepsilon\le 4$)

(ii) $(-\delta,\delta)\subset\left(\sqrt{4-\varepsilon}-2,\sqrt{4+\varepsilon}-2\right)$

$\therefore$ Take $\delta=\min\left(2-\sqrt{4-\varepsilon},\ \sqrt{4+\varepsilon}-2\right)$

Conclusion: if $0<\varepsilon\le 4$,

then $0<|x-2|<\delta \Rightarrow |f(x)-4|<\varepsilon$

For $\varepsilon>4$, take the $\delta=\delta(3)$ obtained from $\varepsilon=3$, then

$0<|x-2|<\delta(3)\Rightarrow |f(x)-4|<3<4<\varepsilon$

~~Suppose $f(x)$ is defined on $(c-\delta, c) \cup (c, c+\delta)$, $\delta > 0$~~

Def: $\lim\limits_{x \to c} f(x) = L$

if for every $\varepsilon > 0$, there exists

(對每個正數ε,皆存在)

a corresponding $\delta > 0$ such that

(一個對應的正數δ,使得下式成立)

"$0 < |x-c| < \delta \Longrightarrow |f(x) - L| < \varepsilon$" (1)

("若(x滿足) $0<|x-c|<\delta$, 則 $|f(x)-L|<\varepsilon$")

Remark:

If we replace (1) by

" $|x-c|<\delta \Longrightarrow |f(x)-L|<\varepsilon$ " (1')

the result is different.

$(1') = (1) + $ " $|f(c)-L|<\varepsilon$ "

(for every $\varepsilon>0$)

That is,

$(1') = (1) + $ " $|f(c)-L|=0$ "

$(1') \Leftrightarrow$ " $\lim_{x\to c} f(x)=L$ " $+$ " $f(c)=L$ "

$\xleftrightarrow{\text{(later)}}$ $f(x)$ is cont. $x=c$

The counter example below shows that (2) cannot be a correct definition for (*) since the example satisfies (2), but not (*)

Wrong Statement:

$$\lim_{x \to c} f(x) = L \qquad (*)$$

if $f(x)$ gets closer to L

as x approaches c

i.e. $|x_2 - c| < |x_1 - c|$

$$\Rightarrow |f(x_2) - L| < |f(x_1) - L| \quad \cdots (2)$$

Counter example: $f(x) = x^2 + 1$, $c = 0$, $L = 0$

$f(x)$ satisfies (2), but $\lim_{x \to c} f(x) \neq L$

Eg 2 If $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$

Prove that $\lim_{x\to c} (2f(x) + g(x)) = 2L + M$

Pf: For any $\varepsilon > 0$, there exists $\delta_1 > 0$, $\delta_2 > 0$ such that

$$0 < |x - c| < \delta_1 \Rightarrow -\frac{\varepsilon}{3} < f(x) - L < \frac{\varepsilon}{3}$$

$$0 < |x - c| < \delta_2 \Rightarrow -\frac{\varepsilon}{3} < g(x) - M < \frac{\varepsilon}{3}$$

take $\delta = \min(\delta_1, \delta_2)$, then

$$0 < |x - c| < \delta \Rightarrow \begin{cases} -2\varepsilon/3 < 2(f(x) - L) < 2\varepsilon/3 \\ -\varepsilon/3 < g(x) - M < \varepsilon/3 \end{cases}$$

$$\Rightarrow -\varepsilon < 2f(x) + g(x) - (2L + M) < \varepsilon \Rightarrow |2f(x) + g(x) - (2L + M)| < \varepsilon$$