

Chap 2 Limit and Continuity

Informal definition: (p81)

Suppose $f(x)$ is defined on $(c-\delta, c) \cup (c, c+\delta)$ for some $\delta > 0$. We say $\lim_{x \rightarrow c} f(x) = L$

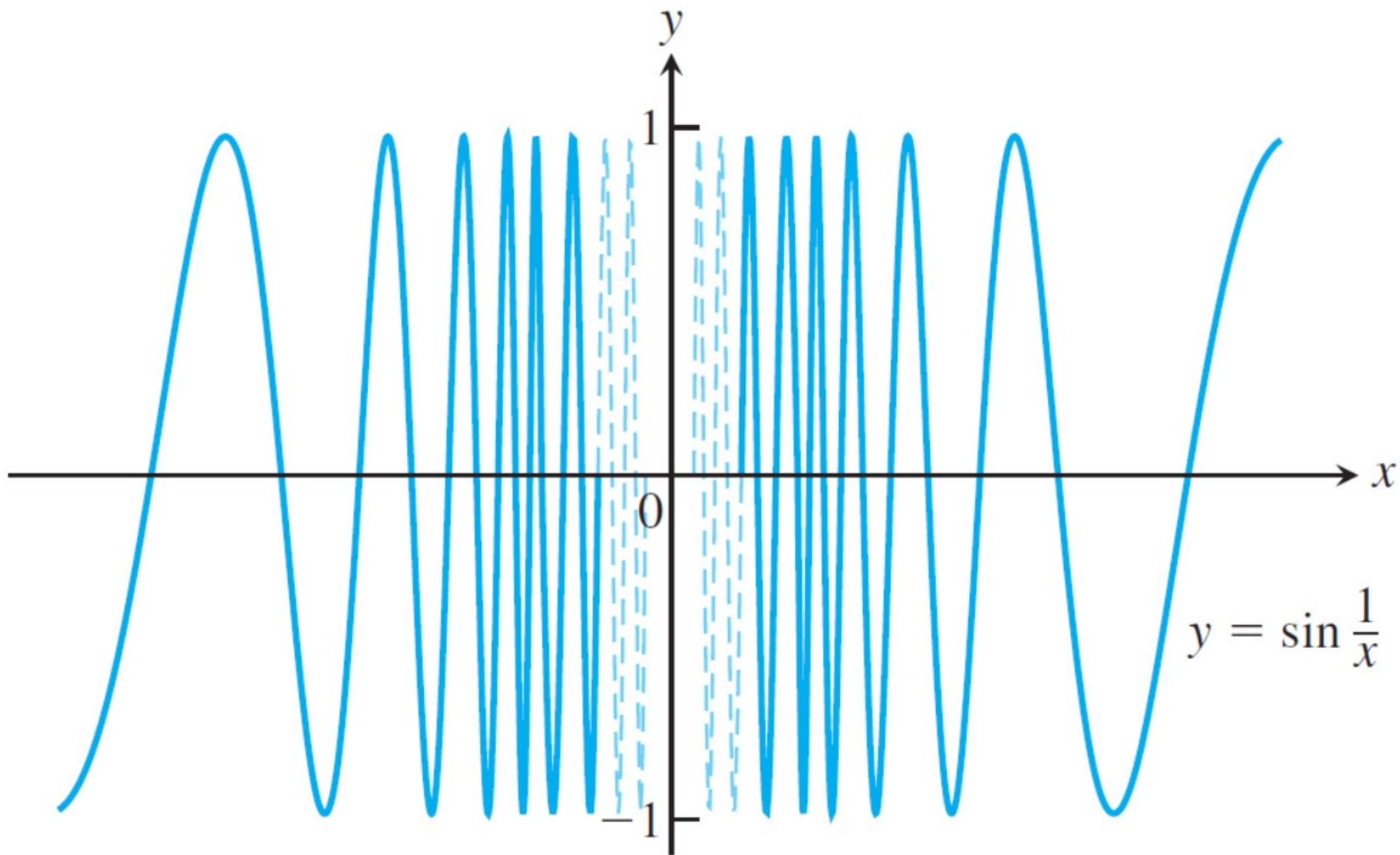
if $f(x)$ is arbitrarily close to L when x is sufficiently close to c .

Eg 1: $\lim_{x \rightarrow 0} f(x)$ does not exist.
(note: $f(0)$ is not relevant)

(i) $f(x) = \frac{x}{|x|}, x \neq 0$ (jump at $x=0$)

(ii) $f(x) = \frac{1}{x}$ or $\frac{1}{x^2}, x \neq 0$ (diverges to $\pm\infty$, unbounded near 0)

(iii) $f(x) = \sin\left(\frac{1}{x}\right), x \neq 0$ (bounded, but oscillatory near 0)
 $= \begin{cases} 0 & x = \frac{1}{n\pi} \\ \pm 1 & x = \frac{1}{(2n \pm \frac{1}{2})\pi} \end{cases}$ (p103) (graph)



Ex 2 (a) $f(x) = x$, $\lim_{x \rightarrow c} f(x) = c$
 (b) $f(x) = k$, $\lim_{x \rightarrow c} f(x) = k$

Thm 1 (Limit laws)

If $\lim_{x \rightarrow c} f(x) = L$, $\lim_{x \rightarrow c} g(x) = M$

Then ① ②: $\lim_{x \rightarrow c} (f(x) \pm g(x)) = L \pm M$

③ $\lim_{x \rightarrow c} (k f(x)) = k \cdot L$

④ $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$

⑤ $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}$ if $M \neq 0$

⑥ $\lim_{x \rightarrow c} (f(x)^n) = L^n$ ($n = \text{positive integer}$)

⑦ $\lim_{x \rightarrow c} f(x)^{\frac{1}{n}} = L^{\frac{1}{n}}$ if $L > 0$

Ex 3

$$\textcircled{a} \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) \quad \left(\begin{array}{l} \text{Limit of} \\ \text{a polynomial} \end{array} \right)$$
$$= c^3 + 4c^2 - 3$$

$$\textcircled{b} \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} \quad \left(\begin{array}{l} \text{Limit of} \\ \text{a rational} \\ \text{function} \end{array} \right)$$
$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$

$$\textcircled{c} \lim_{x \rightarrow -2} \sqrt{4x^2 - 3} = \sqrt{13}$$

("polynomial" + Limit law)

Limits involving quotient
("0" or ("not 0")
0

$$\text{Eg 4. } \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$$

$$\begin{aligned} x^2 - x &= 0 \\ x^2 + x - 2 &= 0 \end{aligned} \quad \text{when } x=1$$

$$\begin{aligned} \Rightarrow x^2 - x &= (x-1) \cdot x \\ x^2 + x - 2 &= (x-1)(x+2) \end{aligned}$$

$$\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \rightarrow 1} \frac{\cancel{(x-1)}(x+2)}{\cancel{(x-1)} \cdot x} = 3$$

$$\text{Eg 5 } \lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{(\sqrt{x^2 + 100} - 10)(\sqrt{x^2 + 100} + 10)}{x^2 (\sqrt{x^2 + 100} + 10)}$$

$$= \lim_{x \rightarrow 0} \frac{\cancel{(a-b)}(a+b) = a^2 - b^2}{\cancel{x^2} (\sqrt{x^2 + 100} + 10)}$$

$$= \frac{1}{\sqrt{0^2 + 100} + 10} = \frac{1}{20}$$

Ex 6 $\lim_{x \rightarrow 1} \frac{x^2 - 1}{(x-1)^2} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x-1)(x-1)}$
(diverges) does not exist

§ Sandwich Theorem

Thm 4. Suppose $g(x) \leq f(x) \leq h(x)$
on $(c-\delta, c) \cup (c, c+\delta)$, $\delta > 0$

If $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$

Then $\lim_{x \rightarrow c} f(x) = L$

$$\text{Eg 7 } \lim_{x \rightarrow 0} x \cdot \sin\left(\frac{1}{x}\right)$$

$$\therefore -1 \leq \sin \frac{1}{x} \leq 1$$

$$-|x| \leq x \sin \frac{1}{x} \leq |x|, \quad x \neq 0$$

$$\lim_{x \rightarrow 0} -|x| = 0 = \lim_{x \rightarrow 0} |x|$$

Sandwich Thm

$$\implies \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

Precise definition of limit

Def: $\lim_{x \rightarrow c} f(x) = L$

if for every $\varepsilon > 0$,
there exists a corresponding $\delta > 0$
such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

~~Suppose $f(x)$ is defined
on $(c-\delta, c) \cup (c, c+\delta)$, $\delta > 0$~~

Def: $\lim_{x \rightarrow c} f(x) = L$

if for every $\varepsilon > 0$, there exists
(對每個正數 ε , 皆存在)

a corresponding $\delta > 0$, such that
(一個對應的正數 δ , 使得下式成立)

$$0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon \quad (1)$$

("若 x 滿足 $0 < |x - c| < \delta$, 則 $|f(x) - L| < \varepsilon$ ")