

Half-Angle substitution

$$\int \frac{P(\cos x, \sin x)}{Q(\cos x, \sin x)} dx, \quad P, Q: \text{polynomials}$$

$$\text{Let } t = \tan \frac{x}{2} \quad dt = \frac{1}{2} \sec^2 \frac{x}{2} dx$$

$$dx = \frac{2dt}{1+t^2}$$

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$$

$$= 2 \tan \frac{x}{2} \cos^2 \frac{x}{2} = \frac{2t}{1+t^2}$$

$$\cos x = 2 \cos^2 \frac{x}{2} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$$

$$\text{Ans} = \int \frac{P\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)}{Q\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)} \frac{2}{1+t^2} dt$$

$$= \int \frac{P_1(t)}{Q_1(t)} dt : \text{partial fraction}$$

(Replace $t = \tan \frac{x}{2}$ in the end)