Midterm 1

Apr 02, 2024: 10:10AM

(I) All Theorems in the textbook can be used directly (unless you are asked precisely to prove that Theorem), just state clearly which Theorem you are using. Results proved in one problem can be used directly in another problem.

(II) All formula that you were asked to memorize, can be used directly unless you are asked explicitly to prove it.

1. (16 pts) Is the integral $\int_0^{\infty} \frac{1}{\sqrt[3]{x+x^2}}dx$ convergent? Find all possible locations of divergence and check out one by one. Do not try to find the anti-derivative.

2. (8 pts) Evaluate $\lim_{x\to 0} \frac{1-\cos^2 x}{\ln(1-x)\,\sin x}$.

3. (12 pts) Find $\sum_{n=1}^{\infty} n(n+1)x^n$ on $|x|<1$ using computational rules for power series (multiplication, differentiation, integration, etc.).

4. (12 pts) Give an approximation of $\int_0^{\frac{1}{2}} \exp(-x^2)dx$ to within 10^{-5}. Give the formula of the approximation, but need not find the numerical value. Explain why the error is less than 10^{-5}.

5. (6+12 pts) (a) Show that the series $\frac{\pi}{3\cdot 1!} - \frac{\pi^3}{3^3\cdot 3!} + \frac{\pi^5}{3^5\cdot 5!} - \cdots$ converges *absolutely*.
   (b) Find the sum of the series in (a). Prove your answer (i.e., why the equality holds).

6. (6+6+6 pts) True or False? Prove it if true, give a counter example if false.
   (a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.
   (b) If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n^2$ converges.
   (c) If $g(x) = f(0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ on $|x|<1$, then $f(x) = g(x)$ on $|x|<1$.

7. (8 pts) Let $f(x) = \frac{e^{-x}\sin x}{\cos 2x}$. Use any method to find $T_{f,0}(x)$ upto x^5 term.

8. (12 pts) Use any method to find $T_{\tan^{-1},0}(x)$. Then find the radius of convergence of this series.