

## Brief solutions to selected problems in homework 15

### 1. Section 16.1: Solutions, common mistakes and corrections:

25.  $y=x^2$   $C_1$   $f=x+\sqrt{y}$   $0 \leq t \leq 1$

$C_1: r(t) = ti + t^2j \Rightarrow ds = \sqrt{1+4t^2}dt$

$C_2: r(t) = \frac{1}{2}i + \frac{1}{2}j \Rightarrow ds = \sqrt{2}dt$

$\Rightarrow \int_C f(x,y) ds = \int_0^1 2t\sqrt{1+4t^2}dt + \int_0^1 \sqrt{2}dt$

$= \frac{1}{3} \int_0^1 (1+4t^2)^{1/2} d(4t^2) + \sqrt{2} \left( \frac{t^2}{2} + \frac{2}{3}t^{3/2} \right) \Big|_0^1$

$= \frac{1}{6} (5\sqrt{5} + 7\sqrt{2} - 1)$

29) In

Figure 1: Solution to Section 16.1, problem 25

### 2. Section 16.3: Solutions, common mistakes and corrections:

Sec 16.3

21.  $\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y} dx + \left( \frac{1}{z} - \frac{x}{y^2} \right) dy + \left( -\frac{y}{z^2} \right) dz$

Find  $f$  satisfies  $\nabla f = \left( \frac{1}{y}, \frac{1}{z} - \frac{x}{y^2}, -\frac{y}{z^2} \right)$

$f_x = \frac{1}{y} \Rightarrow f = \frac{x}{y} + g(y,z) \quad \text{--- (1)}$

$f_y = \frac{1}{z} - \frac{x}{y^2}$ , by (1)  $f_y = -\frac{x}{y^2} + g_y \Rightarrow g_y = \frac{1}{z} \Rightarrow g = \frac{y}{z} + h(z) \quad \text{--- (2)}$

$f_z = -\frac{y}{z^2}$ , by (1,2)  $f_z = 0 + (-\frac{y}{z^2}) + h'(z) \Rightarrow h'(z) = 0 \Rightarrow h(z) = C$  for some constant  $C$ .

$\Rightarrow f = \frac{x}{y} + \frac{y}{z} + C$ . take  $f = \frac{x}{y} + \frac{y}{z}$

$\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y} dx + \left( \frac{1}{z} - \frac{x}{y^2} \right) dy + \left( -\frac{y}{z^2} \right) dz = \left. \frac{x}{y} + \frac{y}{z} \right|_{(1,1,1)}^{(2,2,2)} = 0$

Figure 2: Solution to Section 16.3, problem 21

$$\begin{aligned}
B &= (B_1, B_2, B_3) \\
&= \int_A^B \nabla \sqrt{x^2+y^2+z^2} = \sqrt{B_1^2+B_2^2+B_3^2} - \sqrt{A_1^2+A_2^2+A_3^2} \\
26. \quad A &= (A_1, A_2, A_3) \\
\int_A^B \frac{x dx + y dy + z dz}{\sqrt{x^2+y^2+z^2}} & \quad \text{let } M = \frac{x}{\sqrt{x^2+y^2+z^2}}, N = \frac{y}{\sqrt{x^2+y^2+z^2}}, P = \frac{z}{\sqrt{x^2+y^2+z^2}} \\
\frac{\partial M}{\partial y} &= -\frac{xy}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial N}{\partial x} \quad ; \quad \frac{\partial M}{\partial z} = -\frac{xz}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial P}{\partial x} \\
\frac{\partial N}{\partial z} &= -\frac{yz}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial P}{\partial y} \Rightarrow \text{partial derivatives are continuous} \\
\Rightarrow \mathbb{R}^3 \text{ simply connected} & \Rightarrow \text{conservative} \quad f = \sqrt{x^2+y^2+z^2} \\
& \quad = f(B) - f(A) \\
\Rightarrow F = \nabla f & \text{ \& path independence from } A \text{ to } B \#
\end{aligned}$$

Figure 3: Solution to Section 16.3, problem 26

3. Homework 15, problem 4:

$$\begin{aligned}
F &= \frac{x}{\sqrt{x^2+y^2}} i + \frac{y}{\sqrt{x^2+y^2}} j + 0 k \\
G &= \frac{-y}{x^2+y^2} i + \frac{x}{x^2+y^2} j + 0 k \\
a \quad M_1 &= \frac{x}{\sqrt{x^2+y^2}}, N_1 = \frac{y}{\sqrt{x^2+y^2}}, P_1 = 0 \\
\frac{\partial P_1}{\partial y} &= 0 = \frac{\partial N_1}{\partial z}, \frac{\partial P_1}{\partial x} = 0 = \frac{\partial M_1}{\partial z}, \frac{\partial M_1}{\partial y} = -\frac{xy}{(x^2+y^2)^{\frac{3}{2}}} = \frac{\partial N_1}{\partial x} \\
\Rightarrow F & \text{ satisfies the component test} \\
M_2 &= \frac{-y}{x^2+y^2}, N_2 = \frac{x}{x^2+y^2}, P_2 = 0 \\
\frac{\partial P_2}{\partial y} &= 0 = \frac{\partial N_2}{\partial z}, \frac{\partial P_2}{\partial x} = 0 = \frac{\partial M_2}{\partial z}, \frac{\partial M_2}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial N_2}{\partial x} \\
\Rightarrow G & \text{ satisfies the component test}
\end{aligned}$$

Figure 4: Solution to problem 4(a)

$$\begin{aligned}
 b \quad \nabla f &= F \\
 \frac{\partial f}{\partial x} &= M_1, \quad \frac{\partial f}{\partial y} = N_1, \quad \frac{\partial f}{\partial z} = P_1 \\
 f(x, y, z) &= \sqrt{x^2 + y^2} + g(y, z) \\
 \frac{y}{\sqrt{x^2 + y^2}} + \frac{\partial g}{\partial y} &= \frac{y}{\sqrt{x^2 + y^2}} \Rightarrow \frac{\partial g}{\partial y} = 0 \\
 \Rightarrow f(x, y, z) &= \sqrt{x^2 + y^2} + h(z) \\
 0 + \frac{\partial h}{\partial z} &= 0 \Rightarrow \frac{\partial h}{\partial z} = 0, \quad h(z) = \overset{C}{z + \cancel{C}} \\
 \Rightarrow f(x, y, z) &= \sqrt{x^2 + y^2} + \cancel{z} + C
 \end{aligned}$$

Figure 5: Solution to problem 4(b)

$$\begin{aligned}
 c \quad r(t) &= (\cos t) i + (\sin t) j, \quad 0 \leq t \leq 2\pi \\
 G &= \frac{-y}{x^2 + y^2} i + \frac{x}{x^2 + y^2} j \\
 &= \frac{-\sin t}{\sin^2 t + \cos^2 t} i + \frac{\cos t}{\sin^2 t + \cos^2 t} j \\
 &= (-\sin t) i + (\cos t) j \\
 \frac{dr}{dt} &= (-\sin t) i + (\cos t) j \\
 \oint G \cdot dr &= \oint_C G \cdot \frac{dr}{dt} dt \\
 &= \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt \\
 &= 2\pi \neq 0 \\
 \therefore \oint G \cdot dr &\neq 0 \\
 \therefore G &\text{ isn't conservative by Thm 3.}
 \end{aligned}$$

Figure 6: Solution to problem 4(c)

Problem 4(d): It is easier to explain the idea if we restrict problem 4 in the plane:

Let  $\mathbf{F} = \frac{x}{\sqrt{x^2 + y^2}} \mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}} \mathbf{j}$  and  $\mathbf{G} = \frac{-y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j}$ .

- (a) Show that both  $\mathbf{F}$  and  $\mathbf{G}$  satisfy the component test.
- (b) The natural domain of both  $\mathbf{F}$  and  $\mathbf{G}$  is  $\{(x, y), x^2 + y^2 \neq 0\}$  (that is where  $\mathbf{F}$  and  $\mathbf{G}$  are defined). Show that  $\mathbf{F}$  is conservative in this domain by finding its potential function.
- (c) Show that  $\mathbf{G}$  is NOT conservative in this domain (see Example 5 on p990).
- (d) If given another  $\mathbf{H}$  satisfying the component test in this domain, how do you determine whether  $\mathbf{H}$  is conservative?

**Ans:** It is clear that answers to (a), (b), (c) remain unchanged.

For (d): Suppose  $\mathbf{H}$  satisfies the component test in  $\{(x, y), x^2 + y^2 \neq 0\}$ . Let  $C$  be any simple closed curve, and  $\mathcal{R}$  be the inside of  $C$ .

- (i) If  $(0, 0) \notin \mathcal{R}$ .

In this case,  $\mathcal{R}$  is simply connected. We can apply the 2D version of 'Component Test for Conservative Field' statement on page 988, to conclude that ( $\mathbf{H}$  is conservative, and therefore)

$$\oint_C \mathbf{H} \cdot \mathbf{T} ds = 0 \quad (1)$$

- (ii) If  $(0, 0) \in \mathcal{R}$ , we consider the domain  $\mathcal{R}_\epsilon = \mathcal{R} \setminus \{x^2 + y^2 \leq \epsilon^2\}$ . Note that  $\mathcal{R}_\epsilon$  is simply connected since  $(0, 0) \notin \mathcal{R}_\epsilon$ . Moreover,  $\mathcal{R}_\epsilon = \mathcal{R}_{\epsilon,1} \cup \mathcal{R}_{\epsilon,2}$  as shown Figure 7.

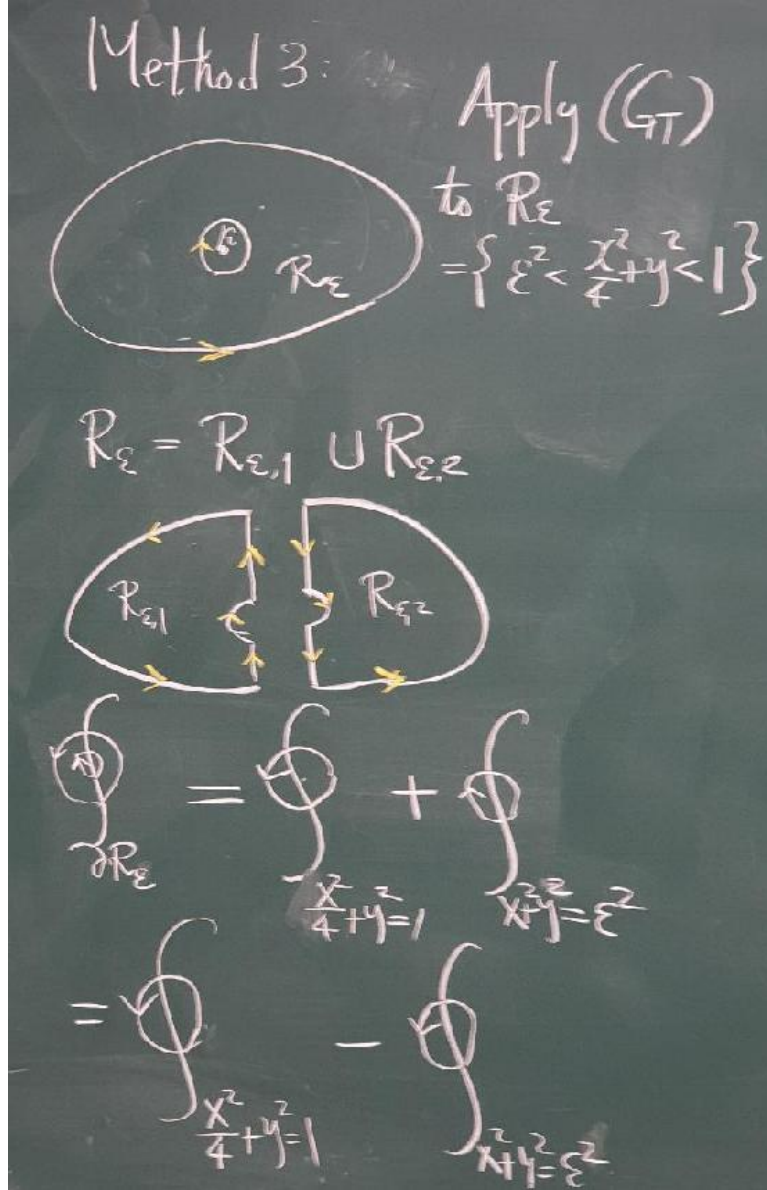


Figure 7:

Using the same argument as in case (a) above (replace  $\mathcal{R}$  by  $\mathcal{R}_{\epsilon,1}$  and  $\mathcal{R}_{\epsilon,2}$ , respectively), we have

$$\oint_{\partial \mathcal{R}_{\epsilon,1}} \mathbf{H} \cdot \mathbf{T} ds = 0, \quad \oint_{\partial \mathcal{R}_{\epsilon,2}} \mathbf{H} \cdot \mathbf{T} ds = 0$$

where  $\partial \mathcal{R}_{\epsilon,i}$  is the boundary of the region  $\mathcal{R}_{\epsilon,i}$ ,  $i = 1, 2$ . As a result, we have

$$\oint_C \mathbf{H} \cdot \mathbf{T} ds = \oint_{C_\epsilon} \mathbf{H} \cdot \mathbf{T} ds \quad (2)$$

where  $C_\epsilon = \{(x, y), x^2 + y^2 = \epsilon^2\}$ . Moreover, it is clear that the line integral in (2) is independent of  $\epsilon > 0$ .

We conclude from the above analysis that,

- (a) If  $\oint_{C_\epsilon} \mathbf{H} \cdot \mathbf{T} ds \neq 0$ , then from Theorem 3 (loop property),  $\mathbf{H}$  is not conservative.
- (b) If  $\oint_{C_\epsilon} \mathbf{H} \cdot \mathbf{T} ds = 0$ , then we conclude from (1), (2) that

$$\oint_C \mathbf{H} \cdot \mathbf{T} ds = 0 \quad (3)$$

for every simple closed curve  $C$ .

If  $C$  is closed but not simple (i.e.  $C$  intersects itself), we can always decompose  $C$  into several simple closed curves (break up at the intersection points and reconnect), it follows that (3) remains valid even if  $C$  is not simple closed.

In summary, we have the following conclusion:

$$\mathbf{H} \text{ is conservative} \iff \oint_C \mathbf{H} \cdot \mathbf{T} ds = 0 \text{ for any closed curve } C \iff \oint_{C_\epsilon} \mathbf{H} \cdot \mathbf{T} ds = 0 \quad (4)$$

The conclusion (4) remains valid in 3D. The argument is similar, with the following replacement of key words:

2D: If  $C$  is simple closed and  $(0,0) \notin \mathcal{R}$ . (3D: If  $C$  does not circle around the  $z$ -axis).

2D: If  $C$  is simple closed and  $(0,0) \in \mathcal{R}$ . (3D: If  $C$  circles around the  $z$ -axis once).

2D:  $C_\epsilon = \{(x,y), x^2 + y^2 = \epsilon^2\}$ . (3D:  $C_\epsilon = \{(x,y,z=0), x^2 + y^2 = \epsilon^2\}$ ).

2D: If  $C$  is not simple closed. (3D: If  $C$  circles around the  $z$ -axis more than once).



4. Problem 5:

Let  $\vec{F} = \frac{1}{\sqrt{x^2+y^2+z^2}} (x, y, z)$ .

- (a) What is the natural domain  $D_F$  of  $\vec{F}$ ?
- (b) Show that  $\vec{F}$  satisfies component test in  $D_F$ .
- (c) Is  $D_F$  simply connected?
- (d) Is  $\vec{F}$  conservative in this domain?

(a)  $D_F = \{(x, y, z) \mid x^2 + y^2 + z^2 > 0\} = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$

(b) 
$$\begin{aligned} \frac{\partial}{\partial y} \left( \frac{x}{\sqrt{x^2+y^2+z^2}} \right) &= -xy(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial x} \left( \frac{y}{\sqrt{x^2+y^2+z^2}} \right) \\ \frac{\partial}{\partial z} \left( \frac{x}{\sqrt{x^2+y^2+z^2}} \right) &= -xz(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial x} \left( \frac{z}{\sqrt{x^2+y^2+z^2}} \right) \\ \frac{\partial}{\partial z} \left( \frac{y}{\sqrt{x^2+y^2+z^2}} \right) &= -yz(x^2+y^2+z^2)^{-\frac{3}{2}} = \frac{\partial}{\partial y} \left( \frac{z}{\sqrt{x^2+y^2+z^2}} \right) \end{aligned}$$

$\therefore \vec{F}$  satisfies component test in  $D_F$

(c)  $D_F$  is simply connected

(d) By (b),  $\vec{F}$  satisfies component test in  $D_F$ .

Also,  $D_F$  is simply connected.

$\therefore \vec{F}$  is conservative in  $D_F$

**Method 2:** By observation (or whatever methods), we know that  $\mathbf{F} = \nabla \sqrt{x^2 + y^2 + z^2}$ , therefore  $\mathbf{F}$  is conservative.