

## Brief solutions to selected problems in homework 12

### 1. Section 14.10: Solutions, common mistakes and corrections:

**14.10**  
 3)  $U = f(P, V, T)$  that obeys  $PV = nRT$   
 4 variables, 2 eqns  $\Rightarrow$  2 dependent, 2 indpt.  
 a)  $\left(\frac{\partial U}{\partial P}\right)_V \rightarrow P$  indpt  $U, T$ : dependent  
 $\rightarrow \frac{\partial U}{\partial P} + \frac{\partial U}{\partial T} \left(\frac{V}{nR}\right)$   
 b)  $\left(\frac{\partial U}{\partial T}\right)_V$   $T, V$ : indpt.  $U, P$ : dependent  
 $= \frac{\partial U}{\partial T} + \frac{\partial U}{\partial P} \left(\frac{nR}{V}\right)$

Figure 1: Solution to Section 14.10, problem 3

① If  $f(x, y, z) = 0$ , then  
 $\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1$   
 For  $\left(\frac{\partial x}{\partial y}\right)_z$ ,  
 $f_x \frac{\partial x}{\partial y} + f_y \frac{\partial y}{\partial y} + f_z \frac{\partial z}{\partial y} = 0$   
 $\Rightarrow \left(\frac{\partial x}{\partial y}\right)_z = -\frac{f_y}{f_x}$   
 Similarly,  $\left(\frac{\partial y}{\partial z}\right)_x = -\frac{f_z}{f_y}$ ,  $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{f_x}{f_z}$

Figure 2: Solution to Section 14.10, problem 9

14.10

12. implies suppose  $x, y$  are independent

$f(x, y, z, w) = 0$ ,  $\frac{\partial f}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = 0$

$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = 0$

$g(x, y, z, w) = 0$ ,  $\frac{\partial g}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial g}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial x} = 0$

$\Rightarrow \frac{\partial g}{\partial x} + \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial x} = 0$

$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$

$y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$ ,  $x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$

$\begin{cases} \frac{\partial f}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial f}{\partial w} \frac{\partial w}{\partial x} = -\frac{\partial f}{\partial x} \\ \frac{\partial g}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial g}{\partial w} \frac{\partial w}{\partial x} = -\frac{\partial g}{\partial x} \end{cases} \Rightarrow \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial w}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial w} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial w} \end{pmatrix}^{-1} \begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial g}{\partial x} \end{pmatrix} = \frac{-\frac{\partial f}{\partial x} \frac{\partial g}{\partial z} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z}}{\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z}}$

Figure 3: Solution to Section 14.10, problem 12

2. Homework 12, problem 2:

14.10

Follow up on Prob 12 (14.10)

Derive a formula for  $\left(\frac{\partial u}{\partial x}\right)_y$

if  $u = U(x, y, z, w)$

$f(x, y, z, w) = 0$

$g(x, y, z, w) = 0$

provided  $f_z g_w - f_w g_z \neq 0$

Since 5 variables  $(u, x, y, z, w)$  and 3 equations  $(U, f, g)$

$\Rightarrow 5 - 3 = 2$  independent variables.

Assume  $x, y$  are independent

$\left(\frac{\partial u}{\partial x}\right)_y = \frac{\partial U}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial U}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial U}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial U}{\partial w} \frac{\partial w}{\partial x}$

$= \frac{\partial U}{\partial x} + \frac{\partial U}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial U}{\partial w} \frac{\partial w}{\partial x}$

$x + y + z = 30$   
 $z = 30 - y - x$

Prob 12

Figure 4: Solution to problem 2

3. Section 15.1: Solutions, common mistakes and corrections:

Problem 36:

Since  $f(x, y)$  is continuous in  $(x, y)$ , it is also continuous in  $x$  for fixed  $y$ , and also continuous in  $y$  for fixed  $x$ . In addition  $\int_c^y f(u, v)dv$  is also continuous in  $u$  for fixed  $y$ .

From Fundamental Theorem of Calculus:

$$\frac{d}{dx} \int_a^x g(u)du = g(x) \text{ provide } g(x) \text{ is continuous.}$$

$$\text{It follows that } F_x(x, y) = \frac{d}{dx} \int_a^x \left( \int_c^y f(u, v)dv \right) du = \int_c^y f(x, v)dv. \text{ Therefore } F_{xy}(x, y) = \frac{d}{dy} F_x(x, y) = \frac{d}{dy} \int_c^y f(x, v)dv = f(x, y).$$

$$\text{From Fubini's Theorem, } F(x, y) = \int_c^y \left( \int_a^x f(u, v)du \right) dv. \text{ Using similar calculation as above, we also have } F_{yx}(x, y) = \frac{d}{dx} F_y(x, y) = \frac{d}{dx} \int_a^x f(u, y)du = f(x, y).$$

4. Section 15.2: Solutions, common mistakes and corrections:

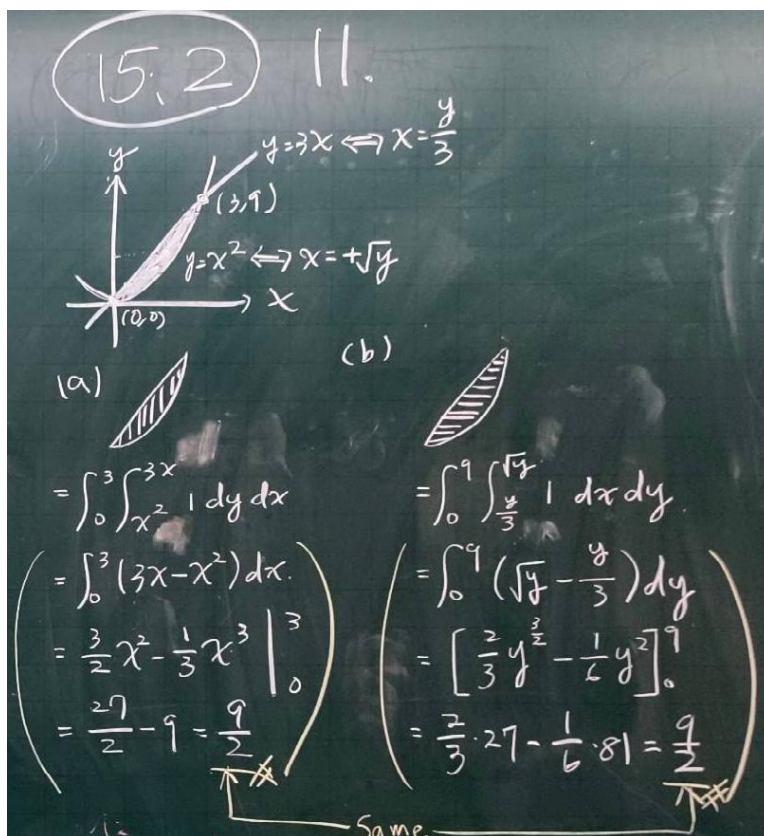


Figure 5: Solution to Section 15.2, problem 11



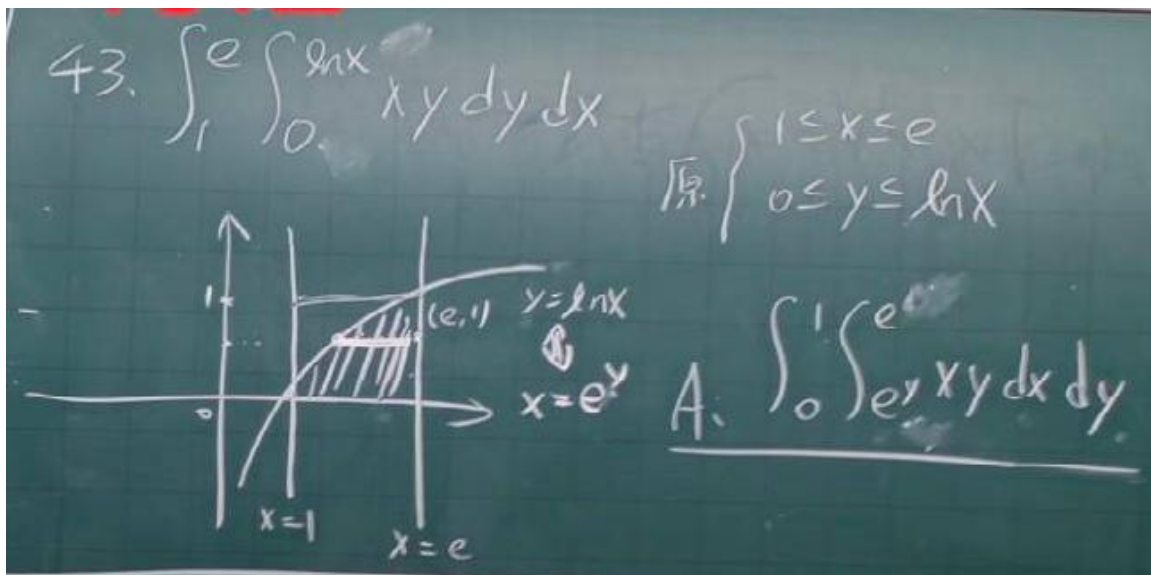


Figure 6: Solution to Section 15.2, problem 43

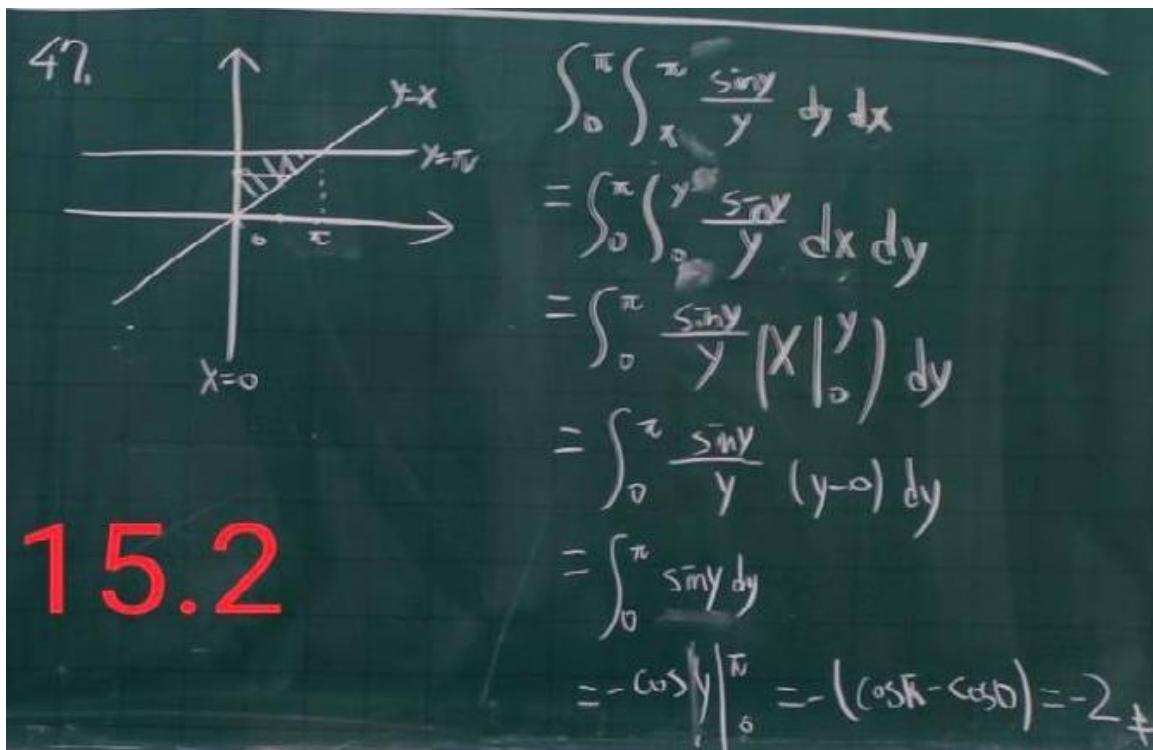


Figure 7: Solution to Section 15.2, problem 47