

Brief solutions to selected problems in homework 10

1. Section 14.6: Solutions, common mistakes and corrections:

17. Let $f: x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0$
 $g: x^2 + y^2 + z^2 = 11$
 $\nabla f = (3x^2 + 6xy^2 + 4y, 6x^2y + 3y^2 + 4x, -2z)$
 $\stackrel{(1,1,3)}{\implies} (3+6+4, 6+3+4, -6) = (13, 13, -6)$
 $\nabla g = (2x, 2y, 2z) \stackrel{(1,1,3)}{\implies} (2, 2, 6)$
 $\vec{N} = \nabla f \times \nabla g = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 13 & 13 & -6 \\ 2 & 2 & 6 \end{vmatrix} = (78+12)\hat{i} + (-78)\hat{j} + 0\hat{k}$
 $= 90\hat{i} - 78\hat{j}$
 $\Rightarrow \text{tangent line } \begin{cases} x = 90t + 1 \\ y = -78t + 1 \\ z = 3 \end{cases}$

Figure 1: Solution to Section 14.6, problem 17

33.
 $f(2, 1) = 3$, $|x-2| \leq 0.1$, $|y-1| \leq 0.1$ $\implies R$
 $f_x(2, 1) = 1$, $f_y(2, 1) = -6$
 $L(x, y) = 3 + (x-2) - 6(y-1)$
 $= x - 6y + 7$
 $f_{xx} = 1, f_{yy} = -6, f_{xy} = 0$ on R
 $\Rightarrow M = 3$ are still continuous
 $|f(x, y) - 3| \leq \frac{1}{2} \times 3 (|x-2| + |y-1|) \leq \frac{1}{2} \times 3 (0.1 + 0.1) = 0.3$

Figure 2: Solution to Section 14.6, problem 33

45.) $f(x, y, z) = xz - 3yz + 2$ at $P(0, 1, 2)$
 $R: |x-1| \leq 0.01 \quad |y-1| \leq 0.01 \quad |z-2| \leq 0.02$

since $f_x = z$, it is clear that f_x is cont. on R

$f(1, 1, 2) = -2 \quad f_x(1, 1, 2) = 2 \quad f_y(1, 1, 2) = -6$
 $f_z(1, 1, 2) = -2$
 since $f_y = -3z$, it is clear that f_y is cont. on R

4 $L(x, y, z) = -2 + 2(x-1) - 6(y-1) - 2(z-2)$
 $= 2x - 6y - 2z + 6$
 since $f_z = x - 3y$, it is clear that f_z is cont. on R

$f_{xx} = 0 \quad f_{yy} = 0 \quad f_{xy} = 0 \quad f_{xz} = 0 \quad f_{yz} = -3$ are still continuous

$M = 3 = \max \{ |f_{xx}|, |f_{yy}|, |f_{xy}|, |f_{xz}|, |f_{yz}| \text{ on } R \}$

$|E(x, y, z)| \leq \frac{1}{2} (3)(0.001 + 0.001 + 0.002)^2 = 0.0024$

Figure 3: Solution to Section 14.6, problem 45

58

show that the curve
 $r(t) = (\sqrt{t}, \sqrt{t}, -\frac{1}{4}(t+3))$
 is normal to the surface
 $x^2 + y^2 - z = 3$ when $t=1$.

① $t=1 \Rightarrow r(1) = (1, 1, -1)$

② $\nabla f = (f_x, f_y, f_z)$
 $= (2x, 2y, -1)$

$\Rightarrow \nabla f(1, 1, -1) = (2, 2, -1) \leftarrow \text{平行}$

③ $v(1) = \frac{dr(t)}{dt} \Big|_{t=1} = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{4} \right)$

Figure 4: Solution to Section 14.6, problem 58

2. Section 14.7: Solutions, common mistakes and corrections:

19. $f(x, y) = 4xy - x^4 - y^4$ $f_{xx} = -12x^2$
 $f_x = 4y - 4x^3$ $f_{xy} = 4$
 $f_y = 4x - 4y^3$ $f_{yy} = -12y^2$

To find the critical point
 Let $f_x = 4y - 4x^3 = 0 \Rightarrow x^3 = y \dots ①$
 $f_y = 4x - 4y^3 = 0 \dots ②$

代入 ② $4x - 4(x^3)^3 = 0$
 $= 4x - 4x^9 = 0$
 $= x(1 - x^8) = 0$
 $= x(1 - x^4)(1 + x^4) = 0$
 $= x(1 - x^2)(1 + x^2)(1 + x^4) = 0$
 $= x(1 + x)(1 - x)(1 + x^2)(1 + x^4) = 0$
 $\Rightarrow x = 0, 1, -1$

critical points are $(0, 0), (1, 1), (-1, -1)$

$D(x, y) = f_{xx}f_{yy} - (f_{xy})^2$
 $= [(-12x^2)(-12y^2)] - 4^2$
 $= 144x^2y^2 - 16$

$D(0, 0) = -16 < 0 \rightarrow$ Saddle point
 $D(1, 1) = 128 > 0$
 $f_{xx} < 0$ } local maximum point
 $D(-1, -1) = 128 > 0$
 $f_{xx} < 0$ } local maximum point

Figure 5: Solution to Section 14.7, problem 19

31. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x=0$, $y=2$, $y=2x$ in the first quadrant.

① 内部: $f_x = 4x - 4 = 0 \Rightarrow x=1$
 $f_y = 2y - 4 = 0 \Rightarrow y=2$
 $\Rightarrow (1, 2)$

② 边界: $f(x, y) = f(0, y) = y^2 - 4y + 1$
 $\frac{d}{dy} f(0, y) = 2y - 4 = 0 \Rightarrow y=2 \Rightarrow (0, 2)$ (C.P.)
 检查端点 on \overline{OA} : $(0, 0)$ (end point)
 $f(0, 2) = -3$
 $f(0, 0) = 1$

③ 边界: \overline{AB} : $(1, 2)$ C.P. $f(1, 2) = -5$
 $(0, 2)$ the end point $f(0, 2) = -3$

④ 边界: \overline{OB} : C.P. $(1, 2)$ $f(1, 2) = -5$
 the end point $(0, 0)$ $f(0, 0) = 1$
 $\Rightarrow \text{Max } f(0, 0) = 1$
 $\text{Min } f(1, 2) = -5$

Figure 6: Solution to Section 14.7, problem 31

Remark: Try gradient analysis (plot ∇f in the region and its tangential component on the boundary), it should be easier to get the conclusion.

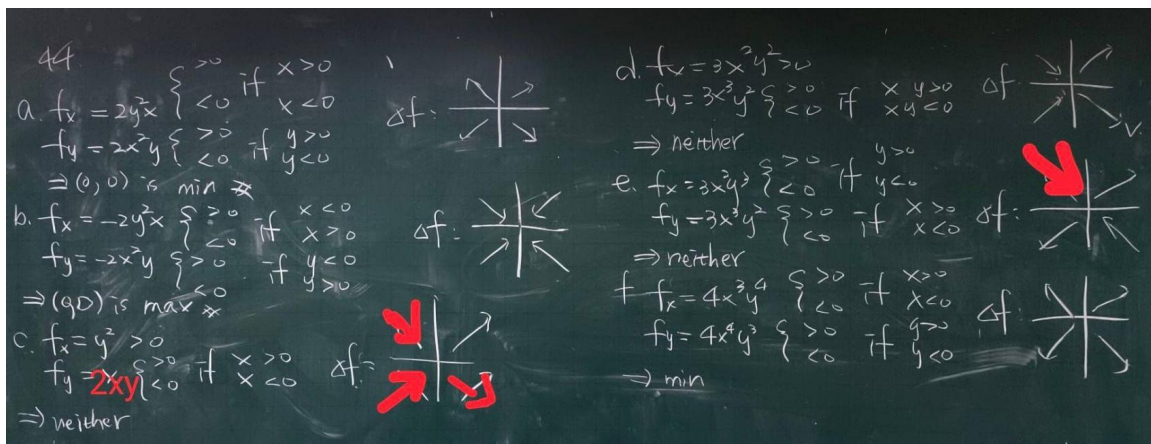


Figure 7: Solution to Section 14.7, problem 44

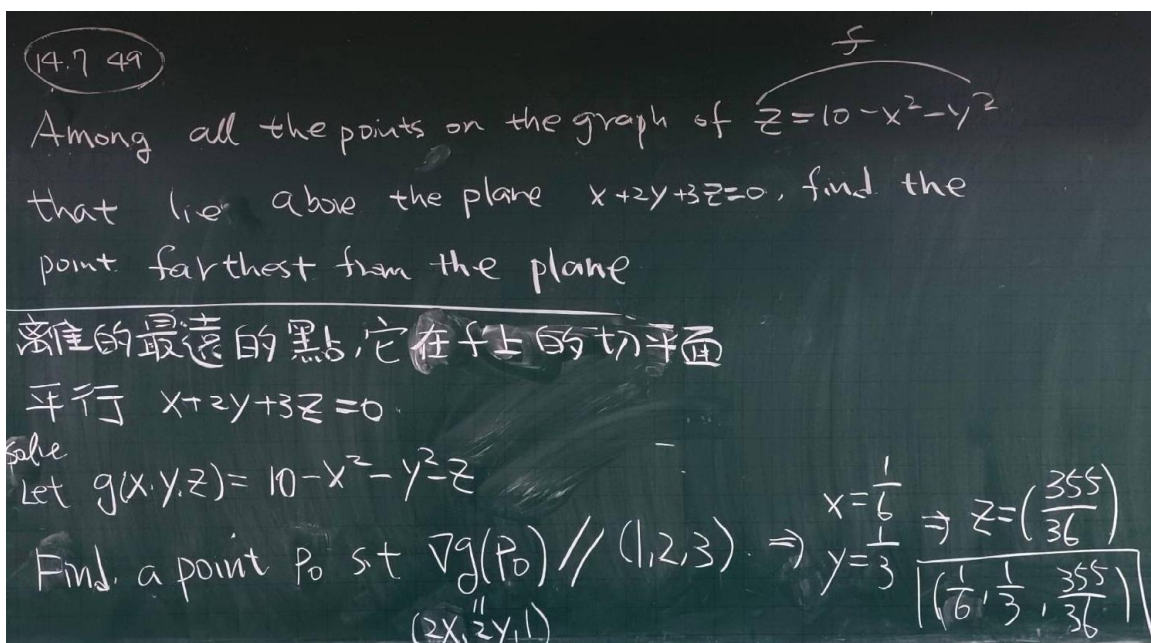


Figure 8: Solution to Section 14.7, problem 49

Remark: Alternative method: Method of Lagrange multiplier (from Section 14.8): Maximize $f(x, y, z) = x + 2y + 3z$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z - 10 = 0$. That is, find the largest value of $k = f(x, y, z)$ on $g(x, y, z) = 0$. Since the larger k is, the larger the distance between the two planes $f(x, y, z) = 0$ and $f(x, y, z) = k$.

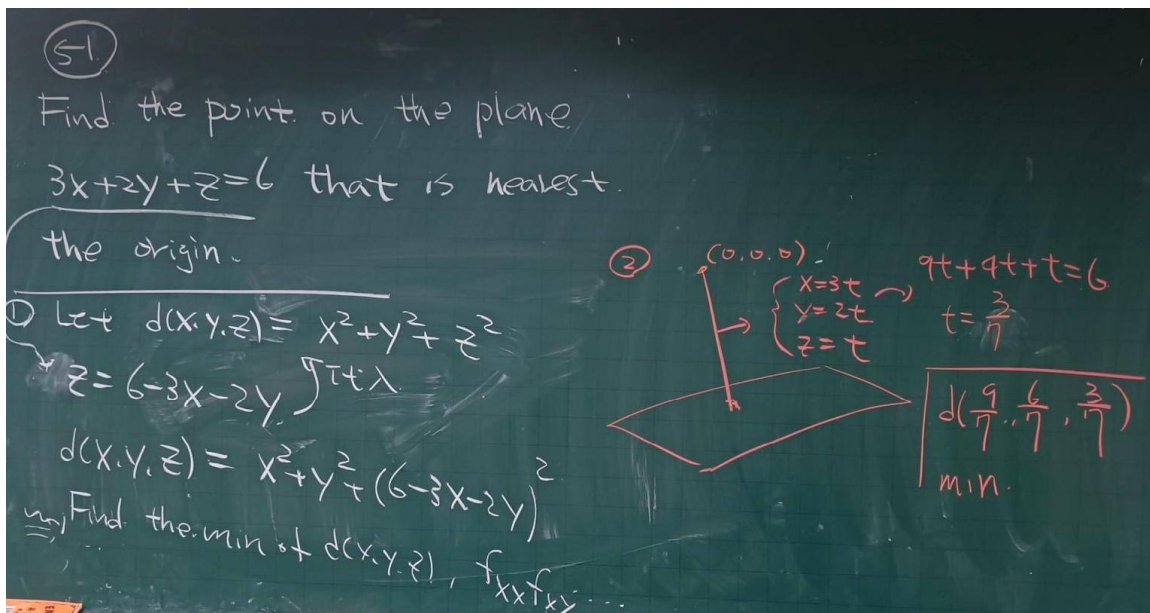


Figure 9: Solution to Section 14.7, problem 51

Remark: Alternative method: Method of Lagrange multiplier (from Section 14.8):
 Minimize $f(x,y,z) = x^2+y^2+z^2$ subject to the constraint $g(x,y,z) = 3x+2y+z-6 = 0$.