

Brief solutions to selected problems in homework 09

1. Section 14.4: Solutions, common mistakes and corrections:

$$\begin{aligned}
 w &= \ln(x^2 + y^2 + z^2), \quad x = ue^v \sin u \\
 &\quad y = ue^v \cos u \quad (u, v) = (-2, 0) \\
 &\quad z = ue^v \\
 \frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial v} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial v} = (W_x, W_y, W_z) \cdot \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right) \\
 &= \left[\frac{2x}{x^2 + y^2 + z^2} \right] \cdot e^v (u \cos u + \sin u) + \left[\frac{2y}{x^2 + y^2 + z^2} \right] \cdot e^v (\cos u - u \sin u) + \left[\frac{2z}{x^2 + y^2 + z^2} \right] \cdot e^v \\
 &= \frac{\sin 2}{2} \cdot 1 \cdot (-2 \cos 2 - \sin 2) - \frac{\cos 2}{2} \cdot 1 \cdot (\cos 2 - 2 \sin 2) - \frac{1}{2} \\
 &= -\left(\frac{1}{2}\right) - \frac{1}{2} = -1 \\
 \frac{\partial w}{\partial v} &= \frac{\sin 2}{2} (u \sin e^v) - \frac{\cos 2}{2} (u \cos e^v) - \frac{1}{2} u \\
 &= \sin^2 2 + \cos^2 2 - \frac{1}{2} \cdot (-2) = 1 + 1 - 2 = 0
 \end{aligned}$$

Figure 1: Solution to Section 14.4, problem 10, method 1

$$\begin{aligned}
 \text{Method 2} \\
 w &= \ln(x^2 + y^2 + z^2) = \ln(2u^2 e^{2v}) \\
 &= \ln 2 + 2 \ln u + 2v \\
 \frac{\partial w}{\partial v} &= 2 \quad \frac{\partial w}{\partial u} = \frac{2}{u}
 \end{aligned}$$

Figure 2: Solution to Section 14.4, problem 10, method 2

43. Suppose f is diff,
 $u = x - y, v = y - z, w = z - x$

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u} \times 1 + \frac{\partial f}{\partial v} \times 0 + \frac{\partial f}{\partial w} \times (-1)$$

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial u} \times (-1) + \frac{\partial f}{\partial v} \times 1 + \frac{\partial f}{\partial w} \times 0$$

$$\frac{\partial g}{\partial z} = \frac{\partial f}{\partial u} \times 0 + \frac{\partial f}{\partial v} \times (-1) + \frac{\partial f}{\partial w} \times 1$$

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = 0 \quad \leftarrow \text{相加.}$$

Figure 3: Solution to Section 14.4, problem 43

Remark: There may be some confusion in the statement of Problem 43. It should have been written as following:

let $f(u, v, w)$ be differentiable and $g(x, y, z) = f(u, v, w)$ where $u = x - y, v = y - z, w = z - x$. Show that $\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z} = 0$

(51)

Fact $F(x) = \int_a^b g(t, x) dt = g(u, x) \cdot \frac{du}{dx} + \int_a^u g_x(t, x) dt$

$\Rightarrow F'(x) = \int_a^b g_x(t, x) dt$

For $F(x) = \int_a^{f(x)} g(t, v) dt$

$\Rightarrow G(u, x) = \int_a^u g(t, x) dt$

$\Rightarrow \frac{d(G(u, x))}{dx} = G_u \cdot \frac{du}{dx} + G_x \cdot \frac{dx}{dx}$

(51) $F(x) = \int_0^{x^2} \sqrt{t^4 + x^3} dt$

$\Rightarrow F'(x) = \sqrt{(x^2)^4 + x^3} \cdot \frac{dx^2}{dx} + \int_0^{x^2} \frac{3x^2}{2\sqrt{t^4 + x^3}} dt$

Figure 4: Solution to Section 14.4, problem 51

Remark: The meaning of $G_u \cdot \frac{du}{dx} + G_x \cdot \frac{dx}{dx}$ is quite clear in the context. Alternatively, one could write it as $\partial_1 G \cdot \frac{du}{dx} + \partial_2 G \cdot \frac{dx}{dx}$ to avoid potential confusion about what $G_x(u(x), x)$ really means.

2. Section 14.5: Solutions, common mistakes and corrections:

14.5.29

$$f(x,y) = x^2 - xy + y^2 - y$$

(a) $\nabla f(1,-1) = 3i - 4j$ $|\nabla f(1,-1)| = 5$, $D_{\vec{u}}f(1,-1) = 5 \Rightarrow \vec{u} = \frac{3}{5}i - \frac{4}{5}j$
($\vec{u} = \frac{\nabla f}{|\nabla f|}$)

(b) $-\nabla f(1,-1) = -3i - 4j$; $\vec{u} = -\frac{3}{5}i + \frac{4}{5}j$
($\vec{u} = -\frac{\nabla f}{|\nabla f|}$)

(c) $D_{\vec{u}}f(1,-1) = 0$, $\vec{u} = \frac{4}{5}i + \frac{3}{5}j$ or $-\frac{4}{5}i - \frac{3}{5}j$
($\vec{u} \perp \nabla f$)

(d) Let $\vec{u} = u_1i + u_2j$ $u_1^2 + u_2^2 = 1$
 $D_{\vec{u}}f(1,-1) = (3i - 4j) \cdot (u_1i + u_2j) = 3u_1 - 4u_2 = 4$

(e) $3u_1 - 4u_2 = -3$, $u_1 = \frac{4u_2 - 3}{3}$
 $\begin{cases} u_2 = 0, u_1 = -1 \\ u_2 = \frac{24}{25}, u_1 = \frac{7}{25} \end{cases}$

$u_2 = \frac{2}{10}$
 $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$
 $\frac{25}{9}$

Figure 5: Solution to Section 14.5, problem 29

35 f is diff.

If $\begin{cases} (D_{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}f)(1,2) = 2\sqrt{2} \\ (D_{(0,-1)}f)(1,2) = -3 \end{cases}$

then find $(D_{(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})}f)(1,2) = ?$

35

$$D_{\vec{u}}f(1,2) = 2\sqrt{2}$$

$$\nabla f = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 2\sqrt{2}$$

$$\nabla f \cdot (0, -1) = -3$$

$$\begin{cases} \frac{1}{\sqrt{2}}f_x + \frac{1}{\sqrt{2}}f_y = 2\sqrt{2} \\ 0 \cdot f_x - f_y = -3 \end{cases}$$

$$\Rightarrow f_y + 3 = 4$$

$$f_x = 1, f_y = 3$$

$$\nabla f = (1, 3)$$

$$(1,3) \cdot \frac{1}{\sqrt{5}}(-1, 2) = \frac{-1}{\sqrt{5}} + \frac{6}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

Figure 6: Solution to Section 14.5, problem 35

① $h(x, y) = \varepsilon_1(x - x_0) + \varepsilon_2(y - y_0)$ with $\lim_{(x, y) \rightarrow (x_0, y_0)} (\varepsilon_1, \varepsilon_2) = (0, 0)$

② $h(x, y) = \varepsilon \sqrt{(x - x_0)^2 + (y - y_0)^2}$ with $\lim_{(x, y) \rightarrow (x_0, y_0)} \varepsilon = 0$

$x - x_0 = \Delta x$, $y - y_0 = \Delta y$

② \Rightarrow ① $\sqrt{(\Delta x)^2 + (\Delta y)^2} = \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \Delta x + \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \Delta y$

$\varepsilon \sqrt{(\Delta x)^2 + (\Delta y)^2} = \varepsilon \Delta x \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} + \varepsilon \Delta y \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$

$\varepsilon_1 = \frac{\varepsilon \Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$, $\varepsilon_2 = \frac{\varepsilon \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \rightarrow h(x, y) = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$\left| \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right|, \left| \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right| < 1$ and $\lim_{(x, y) \rightarrow (x_0, y_0)} \varepsilon = 0 \Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} (\varepsilon_1, \varepsilon_2) = (0, 0)$ # Sandwich thm

Figure 7: Solution to Homework 08, problem 2, part 1

① \Rightarrow ②

Assume $\exists \varepsilon_1, \varepsilon_2$ with $\lim_{(x, y) \rightarrow (x_0, y_0)} (\varepsilon_1, \varepsilon_2) = (0, 0)$

st. $h(x, y) = \varepsilon_1(x - x_0) + \varepsilon_2(y - y_0)$

$\Rightarrow l(x, y) = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$= \left(\frac{\varepsilon_1 \Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} + \frac{\varepsilon_2 \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right) \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Let $\varepsilon =$ #

Since $\left| \frac{\varepsilon_1 \Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right| < |\varepsilon_1|$, $\left| \frac{\varepsilon_2 \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right| < |\varepsilon_2|$

$\lim_{(x, y) \rightarrow (x_0, y_0)} |\varepsilon| \leq \lim_{(x, y) \rightarrow (x_0, y_0)} |\varepsilon_1| + \lim_{(x, y) \rightarrow (x_0, y_0)} |\varepsilon_2| = 0$ #

Figure 8: Solution to Homework 08, problem 2, part 2

3. Problem 3:

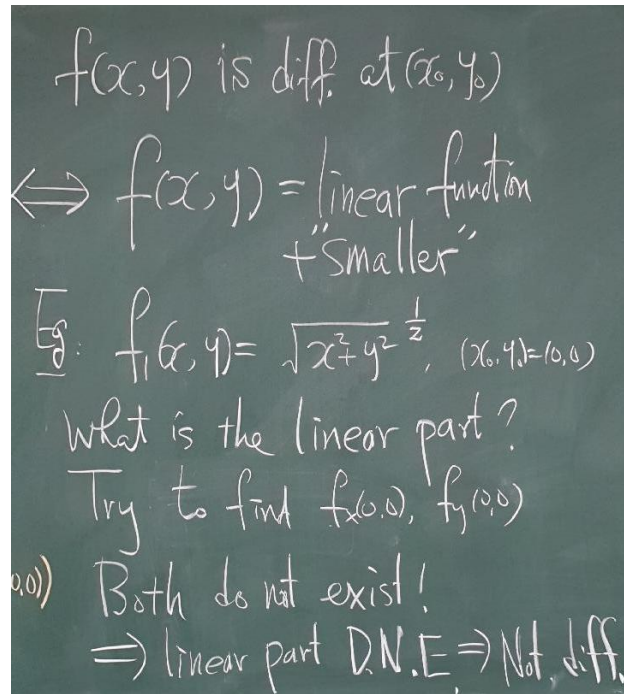


Figure 9: Problem 3

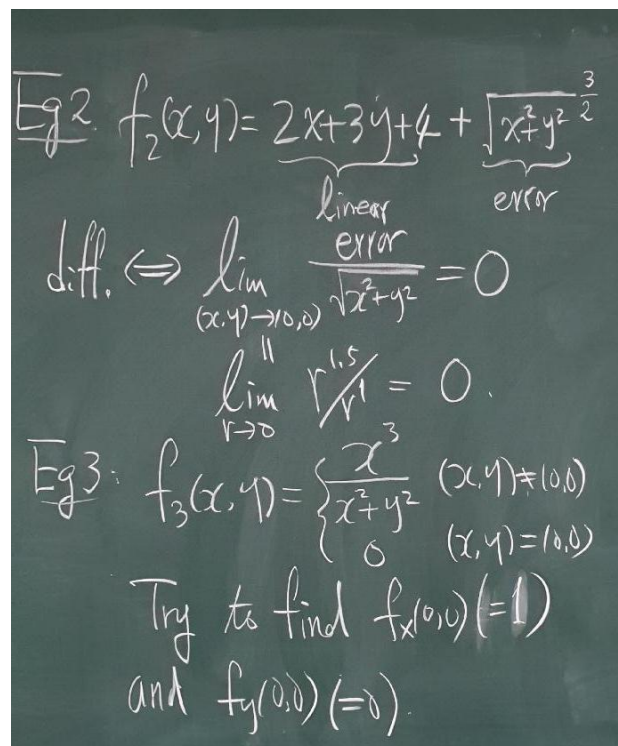


Figure 10: Problem 3, continued

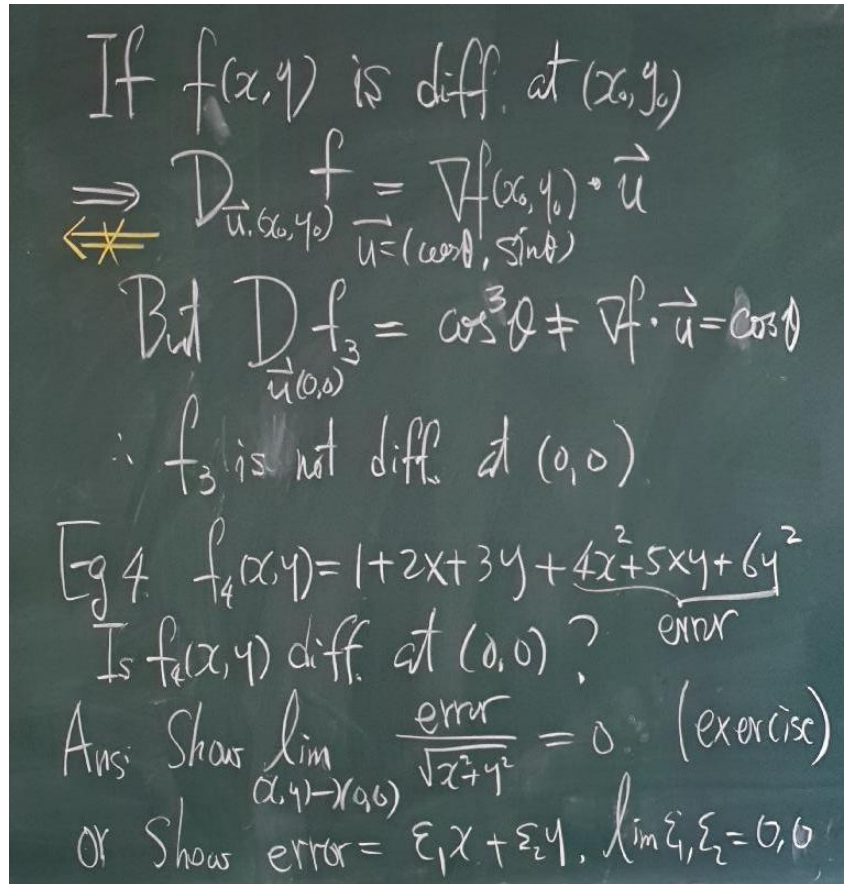


Figure 11: Problem 3, continued

4. (Extra credit, very hard!) True or False?

If $f_x(0,0)$, $f_y(0,0)$ and $D_{(\cos\theta, \sin\theta), (0,0)}f$ all exist and

$$D_{(\cos\theta, \sin\theta), (0,0)}f = f_x(0,0)\cos\theta + f_y(0,0)\sin\theta, \text{ for all } \theta \in [0, 2\pi],$$

then f is differentiable at $(0,0)$.