

Brief solutions to selected problems in homework 08

1. Section 14.3: Solutions, common mistakes and corrections:

14.3 21

Suppose $\int g(t) dt = F(t)$

① $\frac{d(F(x))}{dx} = g(x)$

② $\int_a^b g(t) dt = F(b) - F(a)$

$\frac{\partial}{\partial x} \int_x^y g(t) dt = \frac{\partial}{\partial x} (F(y) - F(x))$

$= 0 - \frac{\partial F(x)}{\partial x} \cdot \frac{\partial}{\partial x} x$

$= -g(x) \cdot 1 = -g(x)$

$\frac{\partial}{\partial y} \int_x^y g(t) dt = \frac{\partial}{\partial y} (F(y) - F(x))$

$= \frac{\partial}{\partial y} (F(y)) - 0 = g(y)$

Figure 1: Solution to Section 14.3, problem 21

60.

$f(x, y) = \begin{cases} \frac{\sin(x^3 + y^4)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ at $(0, 0)$

$A = f(a)$

$\leftarrow \lim_{h \rightarrow 0} \frac{f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h - 0}}{h}$

$= \lim_{h \rightarrow 0} \frac{\frac{\sin(h^3)}{h^2} - 0}{h}$

$= \lim_{h \rightarrow 0} \frac{\sin(h^3)}{h^3} = 1$ (using $\lim_{t \rightarrow 0} \frac{\sin t}{t} = 1$)

$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h - 0}$

$= \lim_{h \rightarrow 0} \frac{\frac{\sin(h^4)}{h^2} - 0}{h}$

$= \lim_{h \rightarrow 0} \frac{\sin(h^4)}{h^3} = 0$ (L.R.)

Figure 2: Solution to Section 14.3, problem 60

69

$$\textcircled{1} - \frac{\partial y}{\partial x} = 0 = V_x \ln u + V \cdot \frac{U_x}{u}$$

$$\textcircled{2} - \frac{\partial y}{\partial x} = 1 = U_x \frac{V}{u} + \frac{V_x}{\ln V}$$

$$\textcircled{1} \times \frac{V}{\ln u} = U_x \frac{V^2}{u} + \frac{V_x V}{\ln V} = 0 \quad \text{---} \textcircled{3}$$

$$\textcircled{2} - \textcircled{3} = 1 = U_x \left(\ln u - \frac{1}{\ln V} \right)$$

$$\Rightarrow U_x = \frac{\ln V}{\ln u \ln V - 1}$$

$$\frac{u \ln V}{(\ln u) u \ln V - u} = \frac{y}{(\ln u) y - u}$$

Figure 3: Solution to Section 14.3, problem 69

72.
$$= \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h}$$

a.
$$\frac{\partial f}{\partial y}(x, 0) = \lim_{h \rightarrow 0} \frac{xh \frac{x^2 + h^2}{x^2 + h^2} - 0}{h} = \lim_{h \rightarrow 0} x \frac{x^2 + h^2}{x^2 + h^2} = x$$

$$\frac{\partial f}{\partial x}(0, y) = \lim_{h \rightarrow 0} \frac{hy \frac{h^2 + y^2}{h^2 + y^2} - 0}{h} = \lim_{h \rightarrow 0} y \frac{h^2 + y^2}{h^2 + y^2} = y$$

b.
$$\frac{\partial^2 f}{\partial y \partial x}(0, 0) = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \Big|_{(0, 0)} = \frac{\partial}{\partial y} (x) \Big|_{(0, 0)} = 1$$

(Method 1)

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \Big|_{(0, 0)} = \frac{\partial}{\partial x} (y) \Big|_{(0, 0)} = 1$$

(Method 2)

$$\frac{\partial^2 f}{\partial y \partial x}(0, 0) = \lim_{h \rightarrow 0} \frac{f_x(0, h) - f_x(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{h - 0}{h} = 1$$

Figure 4: Solution to Section 14.3, problem 72

Remark: Method 2 is the correct way of calculating $f_{xy}(0, 0)$. In contrast, method 1 calculates $\lim_{y \rightarrow 0} f_x(0, y)$. Although they are the same in this problem, they need not be the same in general.

There are examples where $g'(0)$ exists but $\lim_{y \rightarrow 0} g'(y)$ does not exist ($g(y) = y^2 \sin(\frac{1}{y})$ for $y \neq 0$ and $g(0) = 0$).

To complete method 1, one needs to verify that $f_x(0, y)$ is continuous in y , then apply the Mean Value Theorem to $f_x(0, y)$ to conclude that $f_{xy}(0, 0) = \lim_{y \rightarrow 0} f_x(0, y)$ (more complicated).

91.
$$f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^4} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

(1)
$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{0 \cdot h}{h^2 + 0} - 0}{h} = \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

(2)
$$f_y(0, 0) = \lim_{h \rightarrow 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \rightarrow 0} \frac{\frac{0}{0 + h^4} - 0}{h} = 0$$

(3)
$$\lim_{(x, y) \rightarrow (0, 0), x=ky^2} f(x, y) = \lim_{y \rightarrow 0} \frac{ky^4}{(k^2 + 1)y^4} = \frac{k}{k^2 + 1}$$

\therefore Different limit with different k .

$\therefore \lim_{(x, y) \rightarrow (0, 0)} f(x, y)$ doesn't exist

$\Rightarrow f(0, 0)$ isn't continuous at $(0, 0)$.

$\Rightarrow f(0, 0)$ isn't differentiable at $(0, 0)$.

Figure 5: Solution to Section 14.3, problem 91

① $h(x, y) = \varepsilon_1(x - x_0) + \varepsilon_2(y - y_0)$ with $\lim_{(x, y) \rightarrow (x_0, y_0)} (\varepsilon_1, \varepsilon_2) = (0, 0)$

② $h(x, y) = \varepsilon \sqrt{(x - x_0)^2 + (y - y_0)^2}$ with $\lim_{(x, y) \rightarrow (x_0, y_0)} \varepsilon = 0$

$x - x_0 = \Delta x$, $y - y_0 = \Delta y$

② \Rightarrow ① $\sqrt{(\Delta x)^2 + (\Delta y)^2} = \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \Delta x + \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \Delta y$

$\varepsilon \sqrt{(\Delta x)^2 + (\Delta y)^2} = \varepsilon \Delta x \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} + \varepsilon \Delta y \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$

$\varepsilon_1 = \frac{\varepsilon \Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$, $\varepsilon_2 = \frac{\varepsilon \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \rightarrow h(x, y) = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$\left| \frac{\Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right|, \left| \frac{\Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right| < 1$ and $\lim_{(x, y) \rightarrow (x_0, y_0)} \varepsilon = 0 \Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} (\varepsilon_1, \varepsilon_2) = (0, 0)$ # Sandwich thm

Figure 6: Solution to Homework 08, problem 2, part 1

① \Rightarrow ②

Assume $\exists \varepsilon_1, \varepsilon_2$ with $\lim_{(x, y) \rightarrow (x_0, y_0)} (\varepsilon_1, \varepsilon_2) = (0, 0)$

st. $h(x, y) = \varepsilon_1(x - x_0) + \varepsilon_2(y - y_0)$

$\Rightarrow l(x, y) = \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$= \left(\frac{\varepsilon_1 \Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} + \frac{\varepsilon_2 \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right) \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Let $\varepsilon =$ #

Since $\left| \frac{\varepsilon_1 \Delta x}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right| < |\varepsilon_1|$, $\left| \frac{\varepsilon_2 \Delta y}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \right| < |\varepsilon_2|$

$\lim_{(x, y) \rightarrow (x_0, y_0)} |\varepsilon| \leq \lim_{(x, y) \rightarrow (x_0, y_0)} |\varepsilon_1| + \lim_{(x, y) \rightarrow (x_0, y_0)} |\varepsilon_2| = 0$ #

Figure 7: Solution to Homework 08, problem 2, part 2