

Brief solutions to selected problems in homework 06

1. Section 10.9: Solutions, common mistakes and corrections:

$\tan^{-1}x = \int_0^x \frac{1}{1+t^2} dt$
 31. $(\tan^{-1}x)^2$ $\left| \begin{array}{l} (\tan^{-1}x)' = \frac{1}{1+x^2} \\ \tan^{-1}x + C = \int \frac{1}{1+x^2} dx \end{array} \right.$
 $\tan^{-1}x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$
 $(\tan^{-1}x)(\tan^{-1}x) = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots\right)^2$
 $= x^2 - \frac{2}{3}x^4 + \frac{2}{5}x^6 - \frac{2}{7}x^8 + \frac{1}{9}x^4 + \frac{1}{9}x^6 - \frac{2}{15}x^8 + \dots$
 $\dots + \frac{1}{5}x^6 - \frac{1}{15}x^8 + \dots$
 前四項: $x^2 - \frac{2}{3}x^4 + \frac{23}{45}x^6 - \frac{44}{105}x^8 + \dots$

Figure 1: Solution to Section 10.9, problem 31

33. $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$ $= 1 + x + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \dots$ (31. Ho)
 $\Rightarrow e^{\sin x} = 1 + \sin x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$ \Rightarrow to
 $= \sum_{n=0}^{\infty} \frac{(\sin x)^n}{n!}$
 $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
 $\Rightarrow e^{\sin x} = 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) + \frac{1}{2!} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)^2 + \dots$

Figure 2: Solution to Section 10.9, problem 33

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$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}, \quad c \text{ between } x \text{ and } 0$$

$$n=2 \Rightarrow |e^x - (1 + x + \frac{x^2}{2})| \leq \left| \frac{f^{(3)}(0.1)}{3!} (0.1)^3 \right|$$

$$|R_2(x)| \leq \frac{e^{0.1} \cdot 0.1^3}{6}$$

$$\left| \frac{f^{(3)}(c)}{3!} x^3 \right| \approx \underline{\underline{0.000184}}$$

Figure 3: Solution to Section 10.9, problem 41

① **sec10.9 50**

Assume $\exists p \in \mathbb{R}$, s.t.
 $|\pi - p| < 10^{-n}$

\Rightarrow Take $x \in \mathbb{R}$, $p = \pi + x$

$\Rightarrow |x| = |p - \pi| = |\pi - p| < 10^{-n}$

Want to show

$$|p + \sin p - \pi| < 10^{-3}$$

$$|\pi + x + \sin(\pi + x) - \pi| = |x - \sin x|$$

Recall

$$\sin x = 0 + x + 0 + R_2(x)$$

$$= 0 + x + 0 + \frac{\sin'''(\xi)}{3!} x^3, \quad \xi \text{ between } x, 0$$

$$\Rightarrow \sin x - x = \frac{\sin'''(\xi)}{3!} x^3$$

We conclude that

$$|x - \sin x| = \left| \frac{\sin'''(\xi)}{3!} x^3 \right| \leq \frac{1 \cdot |x^3|}{6} < |x^3| = |x|^3$$

by $\leftarrow (10^{-1})^3$

Figure 4: Solution to Section 10.9, problem 50

2. Section 10.10: Solutions, common mistakes and corrections:

Fix $x \in (0, 0.1)$.

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

$$\Rightarrow \int_0^{0.1} \frac{\sin x}{x} dx = \left(x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} \right) \Big|_0^{0.1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$$

$$\Rightarrow \int_0^{0.1} \frac{\sin x}{x} dx < \left| \frac{x^7}{7 \cdot 7!} \right|_{x \in (0, 0.1)} = \left| \frac{(0.1)^7}{7 \cdot 7!} \right| < 10^{-8}$$

See Remark below

Figure 5: Solution to Section 10.10, problem 19

19 估計 $\int_0^{0.1} \frac{\sin x}{x} dx$ (error) $< 10^{-8}$

方法 1

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$$

交錯級數 (alternating series)

方法 1

$$\Rightarrow \left| \frac{\sin x}{x} - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \right) \right| < \left| -\frac{x^6}{7!} \right|$$

$$\Rightarrow \left| \int_0^{0.1} \frac{\sin x}{x} dx - \int_0^{0.1} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \right) dx \right| < \left| -\int_0^{0.1} \frac{x^6}{7!} dx \right|$$

$$< \left| \int_0^{0.1} \frac{(0.1)^6}{7!} dx \right| = (0.1 - 0) \cdot \frac{(0.1)^6}{7!} = \frac{(0.1)^7}{7!} < 10^{-8}$$

方法 2

$$\Rightarrow \int_0^{0.1} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} \right) dx \approx \int_0^{0.1} \frac{\sin x}{x} dx$$

$$\left| \frac{\sin x}{x} - \sum_{n=0}^N (-1)^n \frac{x^{2n}}{(2n+1)!} \right| < \left| (-1)^{N+1} \frac{x^{2(N+1)}}{(2N+3)!} \right|$$

$$\Rightarrow \left| \int_0^{0.1} \frac{\sin x}{x} dx - \int_0^{0.1} \sum_{n=0}^N (-1)^n \frac{x^{2n}}{(2n+1)!} dx \right| < \left| \int_0^{0.1} (-1)^{N+1} \frac{x^{2N+1}}{(2N+3)!} dx \right|$$

$$= \left| \int_0^{0.1} \frac{x^{2N+1}}{(2N+3)!} dx \right| = \left| \frac{x^{2N+2}}{(2N+2)(2N+3)!} \right|_{0.1} = \frac{(0.1)^{2N+2}}{(2N+2)(2N+3)!}$$

$\hookrightarrow \frac{(0.1)^{2N+2}}{(2N+2)(2N+3)!} < 10^{-8} \Rightarrow N \geq 2$

Figure 6: Solution to Section 10.10, problem 19, continued

$$\begin{aligned}
 35. \quad & \lim_{x \rightarrow \infty} x^2 \left(e^{-\frac{1}{x^2}} - 1 \right) \\
 &= x^2 \left(-1 + 1 - \frac{1}{x^2} + \frac{1}{2x^4} - \frac{1}{6x^6} + \dots \right) \\
 & \quad \times -1 + \frac{1}{2x^2} - \frac{1}{6x^4} + \dots \\
 &= \lim_{t \rightarrow 0^+} \frac{1}{t^2} \left(e^{-t^2} - 1 \right) \\
 &= \lim_{t \rightarrow 0} \frac{1}{t^2} \left(-t^2 + \frac{t^4}{2!} - \dots \right) \\
 &= -1
 \end{aligned}$$

Figure 7: Solution to Section 10.10, problem 35

$$\begin{aligned}
 37. \quad & \lim_{x \rightarrow 0} \frac{\ln(1+x^2)}{1-\cos x} = \lim_{x \rightarrow 0} \frac{\frac{2x}{1+x^2}}{\sin x} = 2 \\
 &= \lim_{x \rightarrow 0} \frac{x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \dots}{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)} \\
 &= 2
 \end{aligned}$$

Figure 8: Solution to Section 10.10, problem 37

$$\begin{aligned}
 [\tan^{-1} t]_x^\infty &= \frac{\pi}{2} - \tan^{-1} x \\
 &= \int_x^\infty \frac{1}{1+t^2} dt \\
 &= \int_x^\infty \left(\frac{1}{t^2} - \frac{1}{t^4} + \frac{1}{t^6} - \frac{1}{t^8} + \dots \right) dt \\
 &= \lim_{b \rightarrow \infty} \left(-\frac{1}{t} + \frac{1}{3t^3} - \frac{1}{5t^5} + \frac{1}{7t^7} - \dots \right) \Big|_x^\infty \\
 &= \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \dots
 \end{aligned}$$

Figure 9: Solution to Section 10.10, problem 66

$$\begin{aligned}
 \tan^{-1}(x) &= \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots; \quad x > 1 \\
 [\tan^{-1}(x)]_{-\infty}^x &= \frac{\pi}{2} + \tan^{-1}(x) \\
 &= \int_{-\infty}^x \frac{1}{1+t^2} dt \\
 &= -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots; \quad x < -1 \\
 \Rightarrow \text{When } x < -1, \tan^{-1}(x) &= \frac{-\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \frac{1}{7x^7} - \dots
 \end{aligned}$$

Figure 10: Solution to Section 10.10, problem 66, continued