

Brief solutions to selected problems in homework 04

1. Section 10.3: Solutions, common mistakes and corrections:

Let $d_n = \int_n^\infty \frac{1}{x^{1.1}} dx$.

Goal: Find $N \in \mathbb{N}$

such that $d_N = \int_N^\infty \frac{1}{x^{1.1}} dx \leq 10^{-5}$

$\Rightarrow 10^{-5} \leq N$

Figure 1: Solution to Section 10.3, problem 51

2. Section 10.6: Solutions, common mistakes and corrections:

(28)

$$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n} \rightarrow u_n$$

① $u_n > 0$, $u_n = \frac{1}{n \ln n}$

② $u_{n+1} < u_n$, $> \frac{1}{(n+1) \ln(n+1)} = u_{n+1}$

$$\lim_{n \rightarrow \infty} \frac{1}{n \ln n} = 0$$

Alternating series test \Rightarrow conv.

但加絕對值 $\sum_{n=2}^{\infty} \left(\frac{1}{n \ln n} \right) (\text{div})$

$\int_{\ln 2}^{\infty} \frac{1}{x \ln x} dx$

$\Rightarrow (p=1)$

\Rightarrow conditional conv. #

Figure 2: Solution to Section 10.6, problem 28

(29)
$$\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1}(n)}{n^2 + 1} = a_n$$

Let $f(x) = \frac{\tan^{-1}x}{1+x^2}$ where
 $f(n) = a_n, \forall n \in \mathbb{N}$
 $f'(x) = \frac{1 - (\tan^{-1}x)(2x)}{(1+x^2)^2} \rightarrow 0$

a_n is positive, $a_{n+1} \leq a_n$ $\lim_{n \rightarrow \infty} \frac{\tan^{-1}n}{n^2 + 1} = \lim_{n \rightarrow \infty} \frac{\frac{\pi}{2}}{n^2 + 1} = 0$

Integral Test

$\Rightarrow \int_1^{\infty} \frac{\tan^{-1}x}{x^2 + 1} dx = \lim_{b \rightarrow \infty} \left[\frac{1}{2} (\tan^{-1}(x))^2 \right]_1^b$

$= \lim_{b \rightarrow \infty} \frac{1}{2} (\tan^{-1}(b))^2 - \frac{1}{2} (\tan^{-1}(1))^2$

$= \frac{1}{2} \left(\frac{\pi}{2}\right)^2 - \frac{1}{2} \left(\frac{\pi}{4}\right)^2$

\Rightarrow Absolute convergence

Figure 3: Solution to Section 10.6, problem 29

(49)
$$\left| \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} - \sum_{n=1}^{54} (-1)^{n+1} \frac{1}{n} \right| \leq ?$$

① $a_n > 0$ for all n $\Rightarrow |L - S_4| \leq \left| (-1)^{5+1} \frac{1}{5} \right| = 0.2$

② $a_n > a_{n+1}$ for all n

③ $\lim_{n \rightarrow \infty} a_n = 0$ $|L - S_m| \leq \left| (-1)^{m+2} \frac{1}{m+1} \right|$

$\sum_{n=1}^m (-1)^{n+1} \frac{1}{n}$

Figure 4: Solution to Section 10.6, problem 49

(10.6)
 53 在說至少要到第幾項 (i)
 才會使得下式成立

$$\left| \sum_{n=1}^{\infty} (-1)^n \left[\frac{1}{n^2+3} \right] - \sum_{n=1}^i (-1)^n \frac{1}{n^2+3} \right| \leq 10^{-3}$$

$\textcircled{1} a_n > 0$
 $\textcircled{2} a_n > a_{n+1}, \forall n$
 $\textcircled{3} \lim_{n \rightarrow \infty} a_n = 0, \forall n$

$$\Rightarrow \left| L - \sum_{n=1}^i (-1)^n a_{n+1} \right| \leq \left| (-1)^{i+1} a_{i+1} \right| \leq 10^{-3}$$

$$\left| (-1)^{i+1} \cdot \frac{1}{(i+1)^2+3} \right| \leq 10^{-3}$$

$$\Rightarrow 10^{-3} \geq 30.57 \Rightarrow \boxed{i \geq 3}$$

Figure 5: Solution to Section 10.6, problem 53

3. Section 10.7: Solutions, common mistakes and corrections:

23.

15. $\sum_{n=0}^{\infty} \frac{1}{n^2+3} x^n$

$$\lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{\sqrt{n^2+3}} \right| < 1$$

Root

$$\lim_{n \rightarrow \infty} \left| \frac{1}{\sqrt{n^2+3}} \right| = \frac{1}{\sqrt{1^2+3}} = 1 \Leftrightarrow |x| < 1 \Leftrightarrow -1 < x < 1$$

When $x = -1, \sum_{n=0}^{\infty} (-1)^n \frac{1}{n^2+3}$ conv. by Alternating Series Test

When $x = 1, \sum_{n=0}^{\infty} \frac{1}{n^2+3}$ div.

(a) rad: 1, int. of conv.: $[-1, 1]$
 (b) int. of abs. conv.: $(-1, 1)$
 (c) $x = -1$

~~$|R(x)| < 1$
 $\Rightarrow |x| < 1$
 $\therefore \text{when } |x| < 1, \sum_{n=0}^{\infty} \frac{1}{n^2+3} x^n$ conv.~~

Figure 6: Solution to Section 10.7, problem 15

10.7 19

$$\sum_{n=0}^{\infty} \frac{1^n x^n}{3^n}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{1^{n+1} x^{n+1}}{3^{n+1}}}{\frac{1^n x^n}{3^n}} \right| = \left| \frac{1^{n+1} \cdot x}{3^{n+1}} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{1^{n+1} \cdot x}{3^{n+1}} \right| = \left| \frac{x}{3} \right| = r$$

$$r < 1 \rightarrow \left| \frac{x}{3} \right| < 1 \Rightarrow 3 < x < 3$$

if $x = 3 \rightarrow \sum_{n=0}^{\infty} \frac{1}{3^n} \rightarrow \text{div}$

$x = -3 \rightarrow \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^n}$

$$1. \frac{1}{3^n} = \frac{1}{3^n} > 0$$

$$2. f(x) = \frac{1}{3} \cdot \frac{1}{3} > 0$$

A: $-3 < x < 3$ r not -conditionally

Figure 7: Solution to Section 10.7, problem 19

23.

$$\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n x^n$$

Root Test

$$\lim_{n \rightarrow \infty} \sqrt[n]{\left(1 + \frac{1}{n}\right)^n |x|} = |x| \quad \forall x \in \mathbb{R} \Leftrightarrow -1 < x < 1$$

(a) Radius = 1, ind. of conv. (-1, 1)

converging Series Test (b) conv. (abs) (-1, 1)

(c) No such x.

When $x=1$, by nth-Term Test, $\sum_{n=1}^{\infty} (-1)^n \left(1 + \frac{1}{n}\right)^n$ div.

When $x=-1$, by nth-Term Test, $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$ div.

Figure 8: Solution to Section 10.7, problem 23

4. Homework 04, problem 4:

See page 7 of Lecture 07 for a similar problem. What is the center of the power series for this problem?