

## Brief solutions to selected problems in homework 03

### 1. Section 10.3: Solutions, common mistakes and corrections:

10.3) 28

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$$

Use Int. test.

$$\int_1^{\infty} \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$$

Let  $f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)}$

Let  $\sqrt{x}+1 = u \Rightarrow \frac{1}{2} \frac{1}{\sqrt{x}} dx = du$

$\Rightarrow f(n) \geq 0 \quad \forall n \in \mathbb{N}$   
 $\Rightarrow f$  is cont.  
 $\Rightarrow f$  is decreasing

$$\Rightarrow \int_1^{\infty} \frac{2 du}{\sqrt{2} du} = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)} = \infty$$

Figure 1: Solution to Section 10.3, problem 28

31)

$$\sum_{n=3}^{\infty} \frac{1/n}{\ln(n)\ln(n-1)}$$

$u = \ln n$   $f(x) = \frac{1}{x}$

$\Rightarrow du = \frac{1}{n} dn$

$$\int_3^{\infty} \frac{1/n}{\ln(n)\ln(n-1)}$$

$= \lim_{a \rightarrow \infty} \int_{\ln 3}^a \frac{du}{\ln 3 |u| |u-1|} = (\sec^{-1})'$

$= \sec^{-1} a - \sec^{-1} \ln 3$

$\frac{\pi}{2} - \sec^{-1} \ln 3$  (conv.)

Figure 2: Solution to Section 10.3, problem 31

41.  $\sum_{n=1}^{\infty} \left( \frac{a}{n+2} - \frac{1}{n+4} \right)$   ~~$\sum \frac{a}{n+2}$~~   ~~$-\sum \frac{1}{n+4}$~~

let  $a_n = \frac{a}{n+2} - \frac{1}{n+4} \Rightarrow \text{div}$

$\frac{a}{(n+2)(n+4)} - \frac{1}{(n+2)(n+4)} = \frac{(a-1)n + 4a - 2}{(n+2)(n+4)}$   $\sum a_n + b_n$

if  $a=1, a-1=0 \Rightarrow$   
 let  $b_n = \frac{1}{n^2}$   
 $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = 4a \Rightarrow \sum b_n \text{ conv.} \Rightarrow \sum a_n \text{ conv.}$

if  $a \neq 1$ , let  $c_n = \frac{1}{n}$   
 $\lim_{n \rightarrow \infty} \frac{a_n}{c_n} = a-1 \neq 0 \Rightarrow \sum c_n \text{ div.} \Rightarrow \sum a_n \text{ div.}$

Figure 3: Solution to Section 10.3, problem 41

55.  $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$  set  $(u = \ln x, du = \frac{dx}{x})$   $x = \infty, u = \infty$   
 $x = 2, u = \ln 2$

$= \int_{\ln 2}^{\infty} u^{-p} du = \lim_{b \rightarrow \infty} \left. \frac{u^{-p+1}}{-p+1} \right|_{\ln 2}^b$

$= \lim_{b \rightarrow \infty} \left( \frac{1}{-p+1} \right) \left[ b^{-p+1} - (\ln 2)^{-p+1} \right]$   $\left. \begin{array}{l} \frac{1}{-p+1} (\ln 2)^{-p+1} \quad p > 1 \text{ conv.} \\ \infty \quad p < 1 \text{ div.} \end{array} \right\}$

if  $p=1$   
 $\int_2^{\infty} \frac{dx}{x \ln x} = \lim_{b \rightarrow \infty} (\ln(\ln x)) \Big|_2^b = \infty \text{ div.}$

Figure 4: Solution to Section 10.3, problem 55

2. Section 10.4: Solutions, common mistakes and corrections:

10.4 16.  $\sum \ln(1 + \frac{1}{n^2})$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$  is convergent (p=1)

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{\ln(1 + \frac{1}{n^2})}{\frac{1}{n^2}}$

$= \lim_{n \rightarrow \infty} \ln(1 + \frac{1}{n^2}) \cdot n^2$

$\lim_{n \rightarrow \infty} \ln(1) \cdot n^2 = 0$

By Limit Comparison Test,  $\sum \ln(1 + \frac{1}{n^2})$  is also convergent

but  $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$  and  $\lim_{x \rightarrow \infty} x = \infty$

$\lim_{x \rightarrow \infty} (\frac{1}{x} \cdot x) = 1$

$\lim_{y \rightarrow 0^+} \frac{\ln(1 + y^2)}{y^2} = \frac{0}{0}$

$\lim_{y \rightarrow 0^+} \frac{\frac{2y}{1+y^2}}{2y} = \lim_{y \rightarrow 0^+} \frac{1}{1+y^2} = 1$

Figure 5: Solution to Section 10.4, problem 16

$\sum_{n=1}^{\infty} \frac{1}{2n^{\frac{1}{2}} + n^{\frac{1}{3}}}$

$a_n = \frac{1}{2n^{\frac{1}{2}} + n^{\frac{1}{3}}}$

$b_n = \frac{1}{n^{\frac{1}{2}}}$  (p = 1/2)  $\Rightarrow$  (div.)

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^{\frac{1}{2}}}{2n^{\frac{1}{2}} + n^{\frac{1}{3}}}$

$\therefore b_n$  div

$\therefore a_n$  (div.)

Figure 6: Solution to Section 10.4, problem 17

3. Section 10.4, problem 61:  $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p}$ ,  $p > 1$ .

**Answer:**

**case I:**  $p > 1$ ,  $q \leq 0$ .

Take  $p = 1.5$ ,  $q = -2.3$  for example. Since  $\ln n > 1$  for  $n \geq 3$ , we have

$$\sum_{n=3}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=3}^{\infty} \frac{1}{(\ln n)^{2.3} n^{1.5}} < \sum_{n=3}^{\infty} \frac{1}{n^{1.5}} < \infty.$$

Therefore  $\sum_{n=2}^{\infty} \frac{(\ln n)^{-2.3}}{n^{1.5}}$  converges by the Comparison Test.

The same argument works for any  $p > 1$ ,  $q \leq 0$ . Just replace 1.5 by  $p$  and  $-2.3$  by  $q$ .

**case II:**  $p > 1$ ,  $q > 0$ .

Take  $p = 1.5$ ,  $q = 3.0$  for example. Let  $a_n = \frac{(\ln n)^{3.0}}{n^{1.5}}$ .

Since  $a_n > \frac{1}{n^{1.5}}$  for  $n \geq 3$ , comparing  $\sum_{n=2}^{\infty} a_n$  with  $\sum_{n=2}^{\infty} \frac{1}{n^{1.5}}$  (convergent) leads to no conclusion.

We need to compare  $a_n$  with  $b_n = \frac{1}{n^r}$  by choosing an  $r$  so that  $1 < r < p$ . Therefore we take  $r = \frac{1+p}{2} = 1.25$ ,  $b_n = \frac{1}{n^{1.25}}$ , and apply the Limit Comparison Test:

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n \rightarrow \infty} \frac{(\ln n)^{3.0}}{n^{1.5-1.25}}$$

Instead of applying L'Hôpital's Rule to  $\lim_{n \rightarrow \infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n \rightarrow \infty} \frac{(\ln n)^{3.0}}{n^{1.5-1.25}}$  directly, we notice that

$$\lim_{n \rightarrow \infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n \rightarrow \infty} \left( \frac{\ln n}{n^{\frac{p-r}{q}}} \right)^q = \left( \lim_{n \rightarrow \infty} \frac{\ln n}{n^{\frac{p-r}{q}}} \right)^q = \left( \lim_{n \rightarrow \infty} \frac{\ln n}{n^{\frac{1.5-1.25}{3.0}}} \right)^{3.0}$$

The limit  $\lim_{n \rightarrow \infty} \frac{\ln n}{n^{\frac{p-r}{q}}}$  is easier to compute. By L'Hôpital's Rule:

$$\lim_{n \rightarrow \infty} \frac{\ln n}{n^{\frac{p-r}{q}}} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{\left(\frac{p-r}{q}\right)n^{\frac{p-r}{q}-1}} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{p-r}{q}\right)n^{\frac{p-r}{q}}} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{1.5-1.25}{3.0}\right)n^{\frac{1.5-1.25}{3.0}}} = 0$$

Therefore

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \left( \lim_{n \rightarrow \infty} \frac{\ln n}{n^{\frac{p-r}{q}}} \right)^q = 0$$

Since  $\sum_{n=2}^{\infty} b_n$  converges, we know from the Comparison Test that  $\sum_{n=2}^{\infty} a_n$  also converges.

Again, the same argument works for any  $p > 1$ ,  $q > 0$  and  $1 < r < p$ .



4. Section 10.4, problem 62:  $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p}$ ,  $0 < p < 1$ .

**Answer:** The proof for problem 62 is similar:

**case III:**  $0 < p < 1$ ,  $q \geq 0$ .

Compare it with  $\sum_{n=3}^{\infty} \frac{1}{n^p}$ :

$$\sum_{n=3}^{\infty} \frac{(\ln n)^q}{n^p} > \sum_{n=3}^{\infty} \frac{1}{n^p}$$

Since  $\sum_{n=3}^{\infty} \frac{1}{n^p} = \infty$  for  $0 < p < 1$ , we know by the Comparison Test that  $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p}$  diverges.

**case IV:**  $0 < p < 1$ ,  $q < 0$ .

Compare it with  $\sum_{n=3}^{\infty} \frac{1}{n^r}$ ,  $p < r < 1$  (take  $r = \frac{p+1}{2}$  for example). The rest of the

calculation is similar to case II and leads to the conclusion that  $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p}$  diverges.

5. Section 10.5, problem 25:

Since  $\lim_{n \rightarrow \infty} |a_n| = e^{-3}$ ,  $\implies \lim_{n \rightarrow \infty} a_n \neq 0$ ,  $\implies \sum_{n=1}^{\infty} a_n$  diverges.

6. Section 10.5: Solutions, common mistakes and corrections:

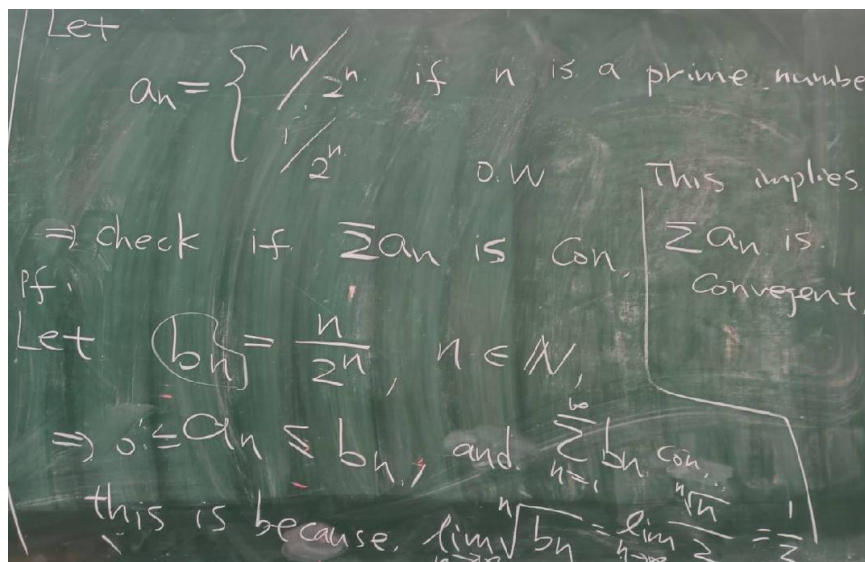


Figure 7: Solution to Section 10.5, problem 65

$$41. \sum_{n=1}^{\infty} \frac{n! \ln n}{n(n+2)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)! \ln(n+1)}{(n+1)(n+3)!} \cdot \frac{n(n+2)!}{n! \ln n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{\ln(n+1)}{n+3} \cdot \frac{n}{\ln n} \right| = 1$$

no conclusion

Figure 8: Solution to Section 10.5, problem 41, part 1

$$n(n+1)(n+2) > n^3$$

$$\Rightarrow \frac{1}{n^3} > \frac{1}{n(n+1)(n+2)}$$

$$0 < \frac{n! \ln n}{n(n+2)!} = \frac{\ln n}{n(n+1)(n+2)} \leq \frac{\ln n}{n^3}$$

$< \frac{1}{n^2}$ , convergent by direct comparison

Figure 9: Solution to Section 10.5, problem 41, part 2