Brief solutions to selected problems in homework 03

1. Section 10.3: Solutions, common mistakes and corrections:

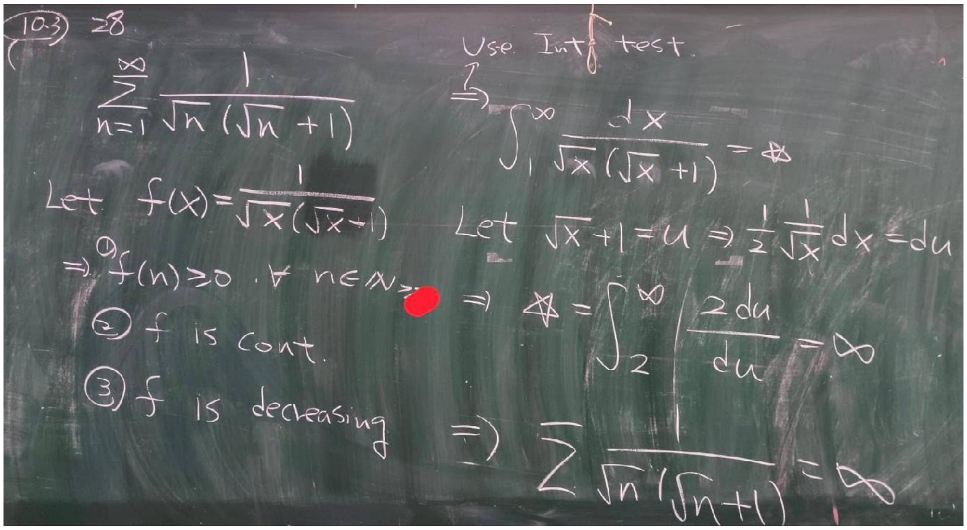

Figure 1: Solution to Section 10.3, problem 28

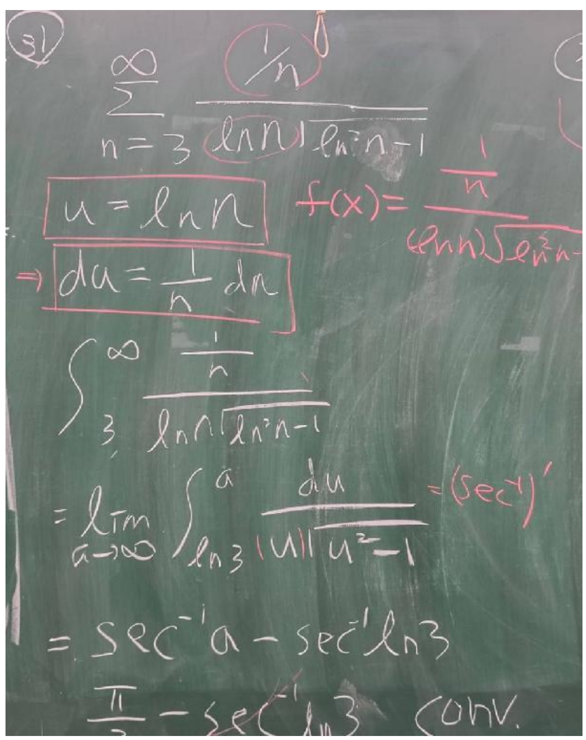

Figure 2: Solution to Section 10.3, problem 31

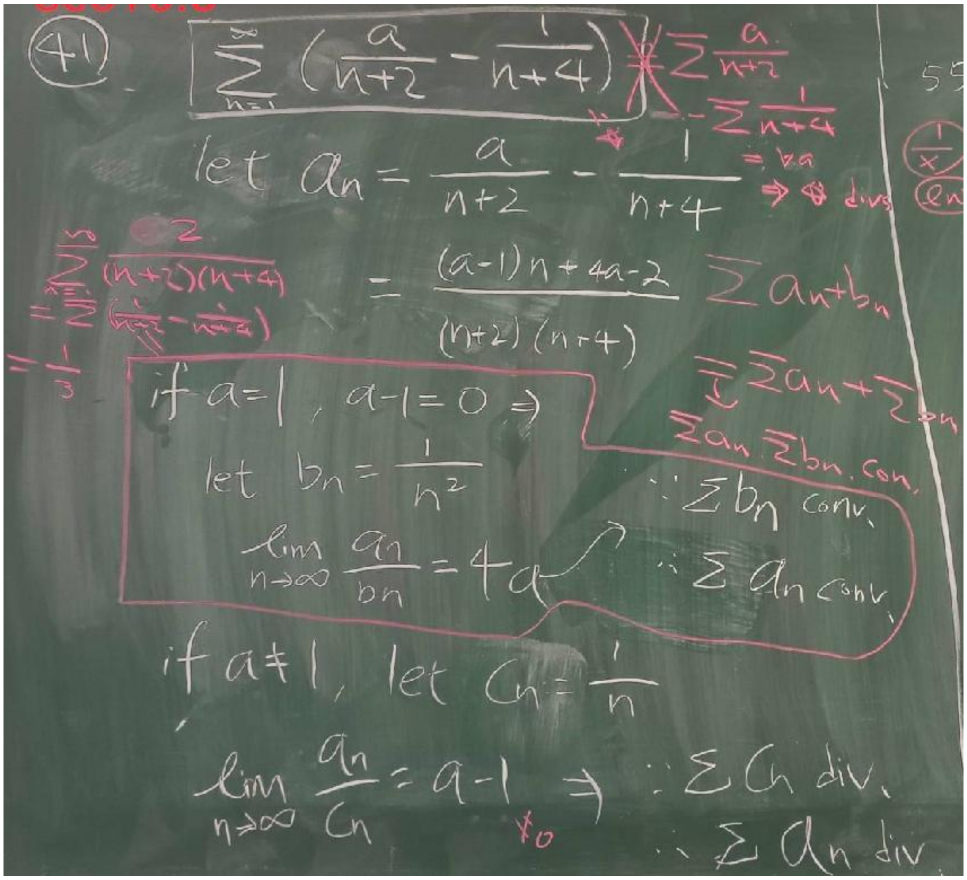

Figure 3: Solution to Section 10.3, problem 41

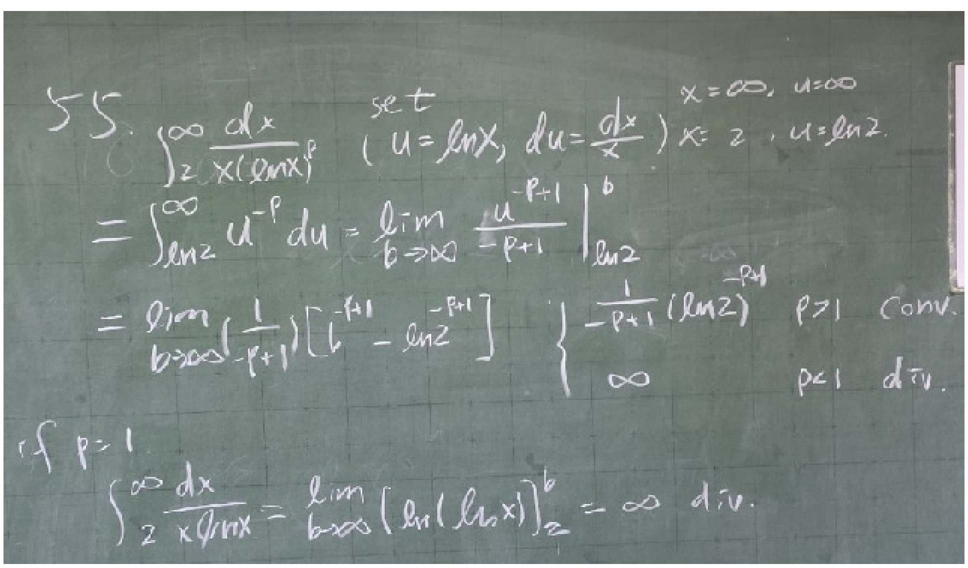

Figure 4: Solution to Section 10.3, problem 55

2. Section 10.4: Solutions, common mistakes and corrections:

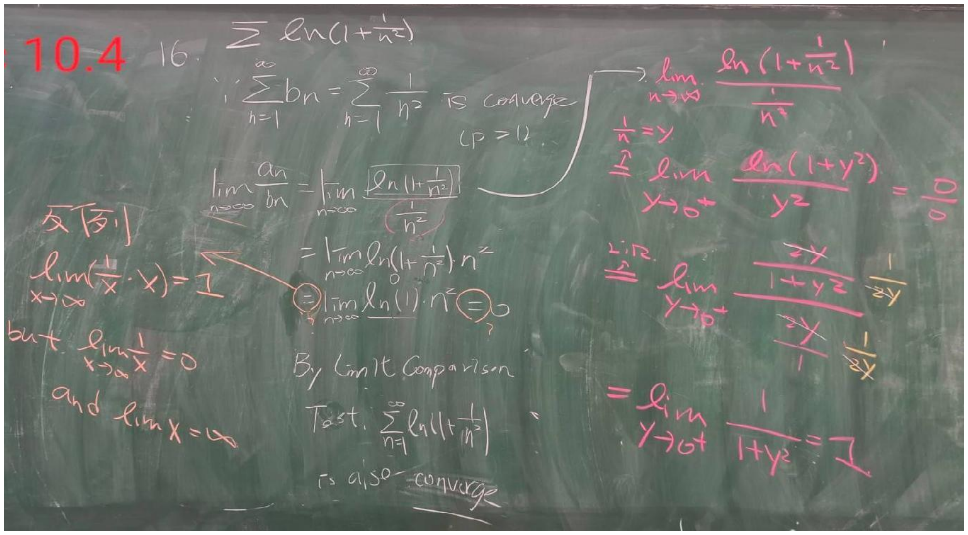

Figure 5: Solution to Section 10.4, problem 16

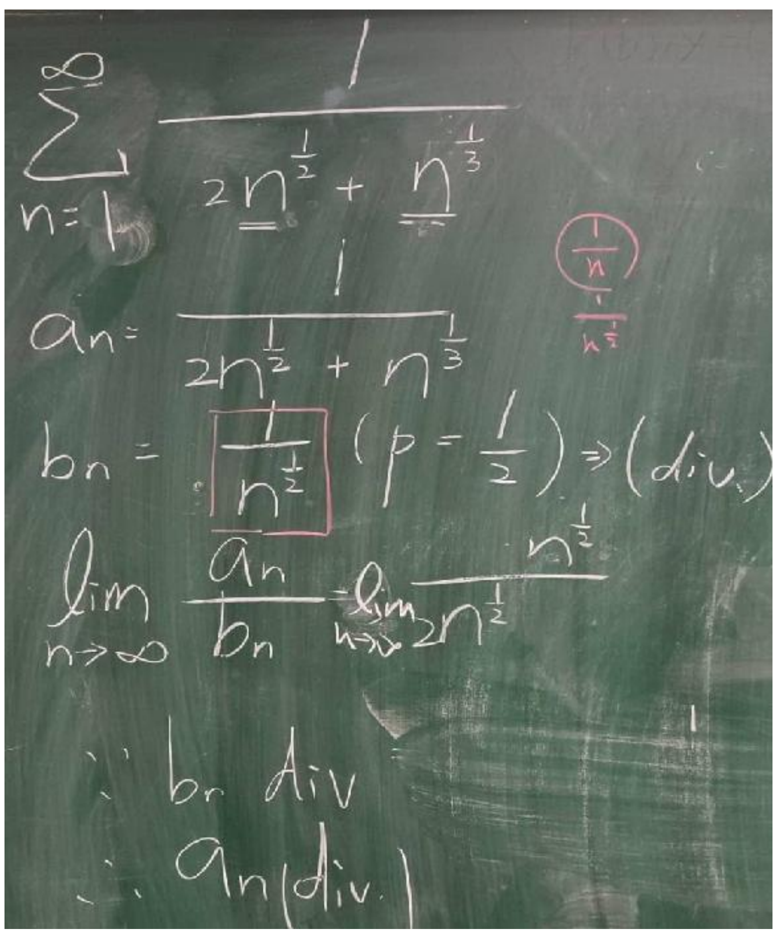

Figure 6: Solution to Section 10.4, problem 17

3. Section 10.4, problem 61: $\sum_{n=2}^{\infty} \frac{(\ln n)^q}{n^p}$, $p > 1$.

Answer:

case I: $p > 1$, $q \le 0$.

Take $p = 1.5$, $q = -2.3$ for example. Since $\ln n > 1$ for $n \ge 3$, we have

$$\sum_{n=3}^{\infty} \frac{(\ln n)^q}{n^p} = \sum_{n=3}^{\infty} \frac{1}{(\ln n)^{2.3} n^{1.5}} < \sum_{n=3}^{\infty} \frac{1}{n^{1.5}} < \infty.$$

Therefore $\sum_{n=2}^{\infty} \frac{(\ln n)^{-2.3}}{n^{1.5}}$ converges by the Comparison Test.

The same argument works for any $p > 1$, $q \le 0$. Just replace 1.5 by p and -2.3 by q.

case II: $p > 1$, $q > 0$.

Take $p = 1.5$, $q = 3.0$ for example. Let $a_n = \frac{(\ln n)^{3.0}}{n^{1.5}}$.

Since $a_n > \frac{1}{n^{1.5}}$ for $n \ge 3$, comparing $\sum_{n=2}^{\infty} a_n$ with $\sum_{n=2}^{\infty} \frac{1}{n^{1.5}}$ (convergent) leads to no conclusion.

We need to compare a_n with $b_n = \frac{1}{n^r}$ by choosing an r so that $1 < r < p$. Therefore we take $r = \frac{1+p}{2} = 1.25$, $b_n = \frac{1}{n^{1.25}}$, and apply the Limit Comparison Test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty} \frac{(\ln n)^{3.0}}{n^{1.5-1.25}}$$

Instead of applying L'Hôpital's Rule to $\lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty} \frac{(\ln n)^{3.0}}{n^{1.5-1.25}}$ directly, we notice that

$$\lim_{n\to\infty} \frac{(\ln n)^q}{n^{p-r}} = \lim_{n\to\infty} \left(\frac{\ln n}{n^{\frac{p-r}{q}}}\right)^q = \left(\lim_{n\to\infty} \frac{\ln n}{n^{\frac{p-r}{q}}}\right)^q = \left(\lim_{n\to\infty} \frac{\ln n}{n^{\frac{1.5-1.25}{3.0}}}\right)^{3.0}$$

The limit $\lim_{n\to\infty} \frac{\ln n}{n^{\frac{p-r}{q}}}$ is easier to compute. By L'Hôpital's Rule:

$$\lim_{n\to\infty} \frac{\ln n}{n^{\frac{p-r}{q}}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{(\frac{p-r}{q}) n^{\frac{p-r}{q}-1}} = \lim_{n\to\infty} \frac{1}{(\frac{p-r}{q}) n^{\frac{p-r}{q}}} = \lim_{n\to\infty} \frac{1}{(\frac{1.5-1.25}{3.0}) n^{\frac{1.5-1.25}{3.0}}} = 0$$

Therefore

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \left(\lim_{n\to\infty} \frac{\ln n}{n^{\frac{p-r}{q}}}\right)^q = 0$$

Since $\sum_{n=2}^{\infty} b_n$ converges, we know from the Comparison Test that $\sum_{n=2}^{\infty} a_n$ also converges.

Again, the same argument works for any $p > 1$, $q > 0$ and $1 < r < p$.

4. Section 10.4, problem 62: $\displaystyle\sum_{n=2}^{\infty}\frac{(\ln n)^q}{n^p}$, $0<p<1$.

   Answer: The proof for problem 62 is similar:

   case III: $0<p<1$, $q\geq 0$.

   Compare it with $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^p}$:

$$\sum_{n=3}^{\infty}\frac{(\ln n)^q}{n^p} > \sum_{n=3}^{\infty}\frac{1}{n^p}$$

   Since $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^p}=\infty$ for $0<p<1$, we know by the Comparison Test that $\displaystyle\sum_{n=2}^{\infty}\frac{(\ln n)^q}{n^p}$ diverges.

   case IV: $0<p<1$, $q<0$.

   Compare it with $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n^r}$, $p<r<1$ (take $r=\frac{p+1}{2}$ for example). The rest of the calculation is similar to case II and leads to the conclusion that $\displaystyle\sum_{n=2}^{\infty}\frac{(\ln n)^q}{n^p}$ diverges.

5. Section 10.5, problem 25:

   Since $\displaystyle\lim_{n\to\infty}|a_n|=e^{-3}$, $\implies$ $\displaystyle\lim_{n\to\infty}a_n\neq 0$, $\implies$ $\displaystyle\sum_{n=1}^{\infty}a_n$ diverges.

6. Section 10.5: Solutions, common mistakes and corrections:

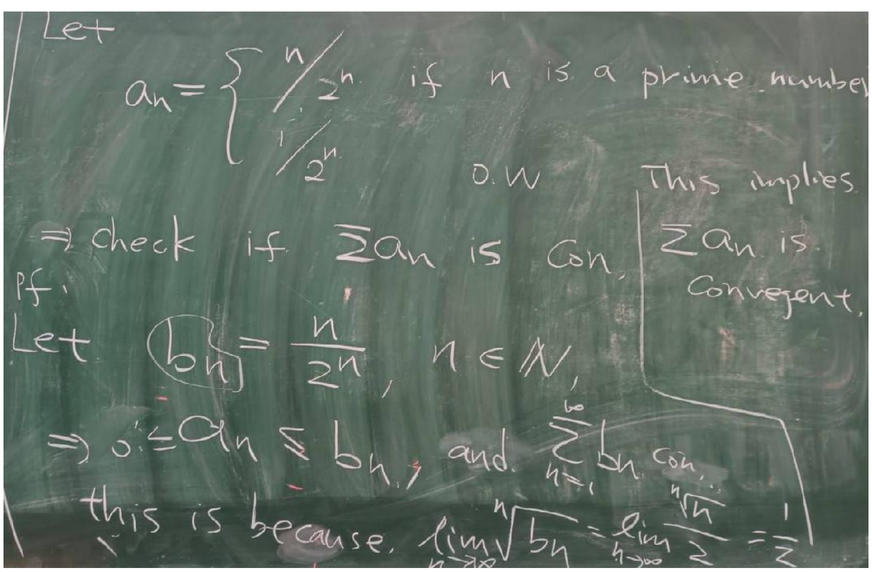

Figure 7: Solution to Section 10.5, problem 65

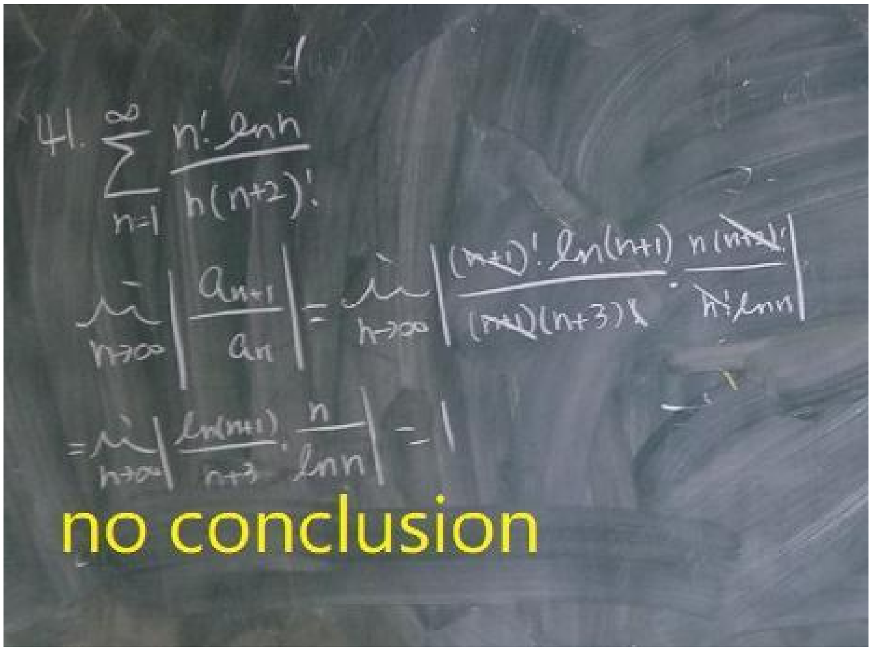

Figure 8: Solution to Section 10.5, problem 41, part 1

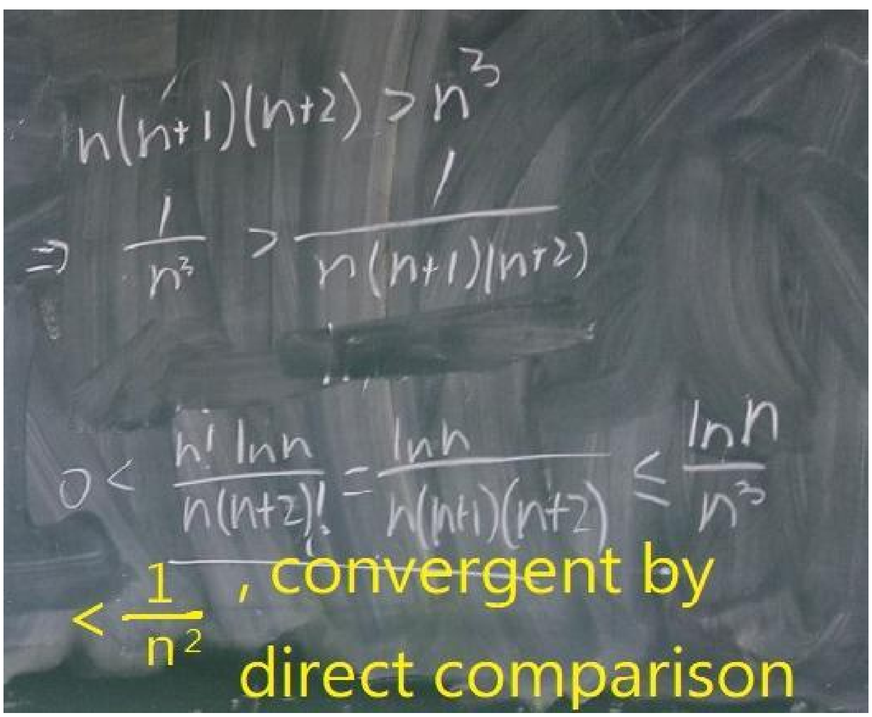

Figure 9: Solution to Section 10.5, problem 41, part 2