

Brief solutions to selected problems in homework 02

1. Section 8.8: Solutions, common mistakes and corrections:

$$\begin{aligned}
 \textcircled{41} \quad & \int_0^\pi \frac{1}{\sqrt{t+\sin t}} dt \\
 0 & \leq \frac{1}{\sqrt{t+\sin t}} \leq \frac{1}{\sqrt{t}} \\
 \int_0^\pi \frac{1}{\sqrt{t}} dt &= \lim_{a \rightarrow 0^+} \int_a^\pi \frac{1}{\sqrt{t}} dt \\
 &= \lim_{a \rightarrow 0^+} 2\sqrt{t} \Big|_a^\pi \\
 &= \lim_{a \rightarrow 0^+} (2\sqrt{\pi} - 2\sqrt{a}) \\
 &= 2\sqrt{\pi} < \infty \\
 \Rightarrow \int_0^\pi \frac{1}{\sqrt{t+\sin t}} dt & \text{converges}
 \end{aligned}$$

Figure 1: Solution to Section 8.8, problem 41

45 even fun.

$$\begin{aligned}
 & \int_{-1}^1 \ln|x| dx \\
 &= 2 \int_0^1 \ln x dx \\
 &= 2 \lim_{a \rightarrow 0^+} \int_a^1 \ln x dx \\
 &= 2 \left(\lim_{a \rightarrow 0^+} \left. x \ln x - x \right|_a^1 \right) \\
 &= 2 \left(-1 - \lim_{a \rightarrow 0^+} a \ln(a - a) \right) \\
 &= -2
 \end{aligned}$$

Figure 2: Solution to Section 8.8, problem 45

Sec 8.8

(55)

$$\int_{\pi}^{\infty} \frac{2+\cos x}{x} dx$$

$$\boxed{\frac{2+\cos x}{x} = \frac{1}{x}} \text{ for } x \in \mathbb{R}_{>0}$$

$$\begin{aligned} \int_{\pi}^{\infty} \frac{1}{x} dx &= \lim_{b \rightarrow \infty} \int_{\pi}^b \frac{1}{x} dx \\ &= \lim_{b \rightarrow \infty} \ln x \Big|_{\pi}^b \\ &= \lim_{b \rightarrow \infty} \ln b - \ln \pi = \infty. \end{aligned}$$

$\therefore \int_{\pi}^{\infty} \frac{1}{x} dx$ diverges & $\frac{2+\cos x}{x} = \frac{1}{x}$ for $x \in \mathbb{R}$.

$\therefore \int_{\pi}^{\infty} \frac{2+\cos x}{x} dx$ diverges.

Figure 3: Solution to Section 8.8, problem 55

(65)

$$\text{(a)} \int_1^2 \frac{dx}{x(\ln x)^p} = \int_0^{\ln 2} \frac{du}{u^p}$$

$$\begin{aligned} \boxed{\ln x = u} \quad &= \begin{cases} \text{con} & \text{if } p < 1 \\ 0.w & \end{cases} \\ \boxed{\frac{1}{x} dx = du} \quad & \end{aligned}$$

(b)

$$\int_2^{\infty} \frac{dx}{x(\ln x)^p}$$

$$= \int_{\ln 2}^{\infty} \frac{du}{u^p} = \begin{cases} \text{con} & \text{if } p > 1 \\ 0.w & \end{cases}$$

Figure 4: Solution to Section 8.8, problem 65

(66)
$$\int_0^\infty \frac{2x}{x^2+1} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{2x}{x^2+1} dx$$

$$= \lim_{b \rightarrow \infty} \left[\ln|x^2+1| \right]_0^b = \lim_{b \rightarrow \infty} \ln|b^2+1| - \ln|1|$$

$$= +\infty$$

Therefore $\int_0^\infty \frac{2x}{x^2+1} dx$ Diverges

$$\lim_{b \rightarrow \infty} \int_{-b}^b \frac{2x}{x^2+1} dx = \lim_{b \rightarrow \infty} \left[\ln|x^2+1| \right]_{-b}^b$$

$$= \lim_{b \rightarrow \infty} \ln|b^2+1| - \ln|b^2+1| = \lim_{b \rightarrow \infty} \ln|1| = 0$$

Odd fun: f
if $f(-x) = -f(x)$
e.g. $f(x) = x^3$



$$\lim_{a \rightarrow \infty} \int_{-a}^a x^3 dx = 0$$

Figure 5: Solution to Section 8.8, problem 66

2. Section 8.8: Homework 02, problem 2:

①
$$\int_e^\infty \frac{dx}{x^{0.5} \ln x}$$

$$f(x) = \frac{x^{-0.5}}{\ln x}$$

$$\text{take } g(x) = \frac{x^{-1}}{\ln x}$$

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} x^{0.5} = \infty$$

$$\int_e^\infty \frac{x^{-1}}{\ln x} dx = \lim_{b \rightarrow \infty} \int_e^b \frac{x^{-1}}{\ln x} dx$$

$$= \lim_{b \rightarrow \infty} (\ln|\ln b| - \ln|\ln e|)$$

$$= \infty$$

$\int_e^\infty g(x) dx$ div.
 $\Rightarrow \int_e^\infty f(x) dx$ div.

Figure 6: Solution to Section homework 02, problem 2

Remark: An alternative approach: Compare with $\int_e^\infty \frac{1}{x^{0.7}} dx$.

3. Section 8.8: Homework 02, problem 3:

$$\begin{aligned}
 & \textcircled{3} \quad \int_0^{\frac{1}{e}} \frac{dx}{x^2 |\ln x|} \\
 & y = \frac{1}{x}, \quad \frac{dy}{dx} = -x^{-2}, \quad dx = \frac{dy}{-y^2} \\
 & \int_{\infty}^{\frac{1}{e}} \frac{y^2 \cdot \frac{1}{|\ln \frac{1}{y}|} \cdot -\frac{dy}{y^2}}{|\ln \frac{1}{y}|} \\
 & = \int_{e}^{\infty} \frac{dy}{|\ln \frac{1}{y}|} = \int_{e}^{\infty} \frac{dy}{-|\ln \frac{1}{y}|} = \int_{e}^{\infty} \frac{dy}{|\ln y|} \\
 & f(x) = \frac{1}{\ln y}, \quad g(x) = \frac{1}{y} \quad (f(x) > g(x)) \\
 & \int_{e}^{\infty} \frac{1}{y} dy \text{ div.} \Rightarrow \int_{e}^{\infty} \frac{dy}{|\ln y|} \text{ div.} \quad \text{※}
 \end{aligned}$$

Figure 7: Solution to Section homework 02, problem 3

Remark: An alternative approach: Compare with $\int_0^{\frac{1}{e}} \frac{1}{x^{1.5}} dx$.

4. Section 10.1: Solutions, common mistakes and corrections:

- Section 10.1, problem 46:

Answer:

Since

$$0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n},$$

and

$$\lim_{n \rightarrow \infty} 0 = 0 = \lim_{n \rightarrow \infty} \frac{1}{2^n},$$

we conclude from the Sandwich Theorem for sequence that

$$\lim_{n \rightarrow \infty} \frac{\sin^2 n}{2^n} = 0.$$

- Section 10.1, problem 59:

Answer:

Since

$$\lim_{n \rightarrow \infty} n^{\frac{1}{n}} = 1, \quad \lim_{n \rightarrow \infty} \ln n = \infty,$$

we have

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n^{\frac{1}{n}}} = \infty \text{ (diverges).}$$

5. Section 10.2: Solutions, common mistakes and corrections:

- Section 10.2, problem 43:

Answer:

From the identity

$$a_n = \frac{40n}{(2n-1)^2(2n+1)^2} = 5 \left(\frac{1}{(2n-1)^2} - \frac{1}{(2n+1)^2} \right),$$

we have

$$s_k = \sum_{n=1}^k a_n = 5 \left(\frac{1}{(2 \cdot 1 - 1)^2} - \frac{1}{(2 \cdot 2 - 1)^2} + \frac{1}{(2 \cdot 2 - 1)^2} - \cdots - \frac{1}{(2 \cdot k + 1)^2} \right) = 5 \left(1 - \frac{1}{(2k+1)^2} \right)$$

Therefore

$$\sum_{n=1}^{\infty} a_n = \lim_{k \rightarrow \infty} s_k = 5.$$

- Section 10.2, problem 61:

Answer:

Since $\lim_{n \rightarrow \infty} a_n = \infty \neq 0$, we conclude from the n -th term test that $\sum_{n=1}^{\infty} a_n$ diverges.

- Section 10.2, problem 65:

Answer:

From the identity

$$a_n = \ln \left(\frac{n}{n+1} \right) = \ln n - \ln(n+1),$$

we have

$$s_k = \sum_{n=1}^k a_n = \ln 1 - \ln 2 + \ln 2 - \ln 3 + \cdots - \ln(k+1).$$

Therefore

$$\sum_{n=1}^{\infty} a_n = \lim_{k \rightarrow \infty} s_k = -\infty \text{ (diverges).}$$

- Section 10.2, problem 78:

Answer:

The geometric series $\sum_{n=0}^{\infty} (\ln x)^n$ converges $\iff |\ln x| < 1 \iff \frac{1}{e} < x < e$.

When $\frac{1}{e} < x < e$, $\sum_{n=0}^{\infty} (\ln x)^n = \frac{1}{1 - \ln x}$.