Partial derivative with Constraint

Example: Find $\frac{\partial w}{\partial \underline{\underline{x}}}$

if $w = x^2 + y^2 + z^2$

and $z = x^2 + y^2$.

Rm: 1 eq + 1 constraint = 2 eq

could be

$$\begin{cases} w = w_1(\underline{\underline{x}}, y) \\ z = z(\underline{\underline{x}}, y) \end{cases} \quad \text{or} \quad \begin{cases} w = w_2(\underline{\underline{x}}, z) \\ y = y(\underline{\underline{x}}, z) \end{cases}$$

$\Rightarrow$ different $\frac{\partial w}{\partial x}$

Need to specify

$$\left(\frac{\partial w}{\partial x}\right)_y \text{ or } \left(\frac{\partial w}{\partial x}\right)_z \quad \text{(different)}$$

indep. var.: (x, y) (x, z)

Here
$$\begin{cases} w_1(x,y) = (x^2+y^2) + (x^2+y^2)^2 \\ z(x,y) = x^2+y^2 \end{cases}$$

$$\begin{cases} w_2(x,z) = z + z^2 \\ y(x,z) = \pm\sqrt{z - x^2} \end{cases}$$

Eg. Find $\left(\frac{\partial w}{\partial x}\right)_y$ at $(x, y, z) = (2, -1, 1)$

if $w = x^2 + y^2 + z^2$

and $z^3 - xy + yz + y^3 = 1$

Ans: $w = w(x, y)$, $z = z(x, y)$

$$w_x = 2x + 2z z_x \quad — ①$$

$$3z^2 z_x - y + y z_x = 0 \quad — ②$$

$$② \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y (2,-1,1) = \frac{y}{3z^2 + y}\Big|_{(2,-1,1)} = \frac{-1}{2}$$

$$\stackrel{①}{\Longrightarrow} \left(\frac{\partial w}{\partial x}\right)_y (2,-1,1) = 4 + 2 \cdot \frac{-1}{2} = 3$$

Eg Find $\left(\frac{\partial w}{\partial x}\right)_{y,z}$ if $w = x^2 + y - z + \sin t$

and $x + y = t$

indep. var.: x, y, z

dep. var.: $w(x,y,z)$, $t(x,y,z)$

$\Rightarrow w(x,y,z) = x^2 + y - z + \sin(x+y)$

$$\left(\frac{\partial w}{\partial x}\right)_{y,z} = 2x + \cos(x+y)$$

Eg Find $\left(\frac{\partial z}{\partial x}\right)_y$ if

$$f(x, y, z, w) = 0$$

and $g(x, y, z, w) = 0$

Ans: $z = z(x, y)$, $w = w(x, y)$

$$\Rightarrow \left(\frac{\partial}{\partial x}\right)_y \Rightarrow \begin{cases} f_x + f_z z_x + f_w w_x = 0 \\ g_x + g_z z_x + g_w w_x = 0 \end{cases}$$

2 unknowns (z_x, w_x), 2 eqns

$\Rightarrow$ Solve 2×2 linear systems.

Eg: Find $\left(\frac{\partial(PV)}{\partial T}\right)_{n,V}$

if $PV = nRT$, R = const.

Sol: indep. var. T, n, V

$$P = P(T, n, V)$$

$$\frac{\partial(P(T,n,V)\cdot V)}{\partial T} = P_T V$$

$$P = \frac{nRT}{V}, \quad \left(\frac{\partial P}{\partial T}\right)_{n,V} = \frac{nR}{V}$$

$$\text{Ans} = \left(\frac{\partial P}{\partial T}\right)_{n,V} \cdot V = nR$$

Rm $\left(\frac{\partial(PV)}{\partial T}\right)_n$ $PV = nRT$

$\Rightarrow P = f(T, n)$ $V = g(T, n)$

$$f(T, n)\, g(T, n) = nRT$$

Suprisingly, $\left(\frac{\partial(PV)}{\partial T}\right)_n$

does not depend explicitly on f or g. (HW)