Def: $\vec{F}$ is conservative if $\int_C \vec{F}\cdot d\vec{r}$ only depends on starting point and end point of C. (and not on the rest of C).

Thm 1 (assumptions)

$\vec{F} = \nabla f \Longrightarrow \vec{F}$ is conservative.

Thm 2 (If D is connected)

" $\Longleftarrow$ " (proof: textbook)

Thm 3

$\vec{F}$ is conservative in D

$\Leftrightarrow \oint_C \vec{F} \cdot d\vec{r} = 0$ for any closed curve $C \subseteq D$.

<u>Component test</u> (and 2D version)

Let $\vec{F} = (M(x,y,z), N(x,y,z), P(x,y,z))$ with M, N, P and their partial derivative continuous in D.

Then

(i) $\vec{F}$ is conservative $\Rightarrow \begin{cases} M_y = N_x \\ N_z = P_y \quad (*) \\ P_x = M_z \end{cases}$

(ii) " $\Leftarrow$ " if D is simply connected

Question:

When is $\vec{F}$ conservative?

i.e. Can we find the potential f?

Eg 1. $\vec{F} = (e^x \cos y + yz,\ xz - e^x \sin y,\ xy + z)$

Is $\vec{F}$ conservative?

Sol: Check component test first

Is (*) true? No $\Longrightarrow$ Not conservative

Yes $\Longrightarrow$ try to find f

maybe conservative, maybe not

check. $M_y = -e^x \sin y + z = N_x$

$N_z = x \qquad = P_y$

$P_x = y \qquad = M_z$

(1) F is defined everywhere,

\+ so $D = R^3$ is simply connected

(2)(*) holds $\Rightarrow$ f ~~may probably~~ must exist!

(from component test (ii))

How to find f?

$$f_x = e^x \cos y + yz$$

$$\Rightarrow f = \int (e^x \cos y + yz)\,dx$$

$$= e^x \cos y + xyz + C_1(y,z)$$

Similarly

$$f = xyz + e^x \cos y + C_2(x,z)$$

$$f = xyz + \frac{z^2}{2} + C_3(x,y)$$

$$\Rightarrow f(x,y,z)$$

$$= xyz + e^x \cos y + \frac{z^2}{2} \text{ will do!}$$

Eg2. $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right)$

Is $\vec{F}$ conservative?

Sol: $M_y = N_x = \frac{y^2 - x^2}{(x^2+y^2)^2}$

$\therefore$ (*) holds!

$\vec{F}$ is defined on $D = \mathbb{R}^3 \setminus z$ axis

D is not Simply connected.

Try to find $f \Rightarrow$ does not work

Try to prove $\vec{F}$ is not conservative.

$\Rightarrow$ Use Thm 2

Take C: $\vec{r}(t) = (a\cos t, a\sin t, 0)$

with $a > 0$ fixed, $0 \leq t \leq 2\pi$

$\vec{r}'(t) = (-a\sin t, a\cos t, 0)$

$$\oint_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} \left(\frac{-\sin t}{a}, \frac{\cos t}{a}, 0\right)\cdot(-a\sin t, a\cos t, 0)\,dt$$

$$= 2\pi$$

From Thm 2 $\Rightarrow$ $\vec{F}$ is not conservative.

Eg 3: $\vec{F} = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 0\right)$ conservative?

Sol: $N_y = M_x = \frac{-1}{2}\left(xy(x^2+y^2)^{\frac{-3}{2}}\right)$

$\therefore$ f may exist. Take same C as in Eg 2.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (c, s, 0)\cdot(-as, ac, 0)\,dt = 0$$

Try to find f $\Rightarrow$ $f(x,y) = \sqrt{x^2+y^2} + C$

$\therefore$ $\vec{F}$ is conservative!