Line integrals

Eg 1 Evaluate $\int_C f(x,y,z)\,ds$

where $f = x - 3y^2 + z$

C = line segment <u>between</u>

$(0,0,0)$ and $(1,1,1)$.

<u>Sol</u>: <u>Step 1</u>: Find $\vec{r}(t)$ for C

such as $\vec{r}(t) = (t,t,t),\ 0 \le t \le 1$

<u>Step 2</u> $ds = |\dot{\vec{r}}(t)|\,dt = |(1,1,1)|\,dt$

<u>Step 3</u>: $Ans = \int_{t=0}^{1} (t - 3t^2 + t)\sqrt{3}\,dt = 0$

Rm $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

Rm: the value of $I = \int_C f(x,y,z)\,ds$

is independent of the parameter

$\vec{r}(t)$. For example

$\vec{r}(t) = (t^2, t^2, t^2)$, $0 \le t \le 1$

or $\vec{r}(t) = (1-t, 1-t, 1-t)$, $0 \le t \le 1$

all give the same value of

I as long as $\vec{r}(t)$ is

a correct parametrization

for C.

Eg2: $f(x,y,z) = x - 3y^2 + z$

$C = C_1 \cup C_2$ (two line segments)

$C_1 = \overline{(0,0,0)\ (1,1,0)}$

$C_2 = \overline{(1,1,0)\ (1,1,1)}$

Sol: $C_1 : \vec{r_1}(t) = (t, t, 0),\ 0 \le t \le 1$

$C_2 : \vec{r_2}(t) = (1, 1, t),\ 0 \le t \le 1$

$$\int_C = \int_{C_1} + \int_{C_2}$$

$$= \int_0^1 (t - 3t^2) \overset{\left|\frac{d\vec{r_1}}{dt}\right|}{\sqrt{2}}\, dt + \int_0^1 (-2+t) \cdot \overset{\left|\frac{d\vec{r_2}}{dt}\right|}{1} \cdot dt$$

$$= \frac{-\sqrt{2}}{2} + \frac{-3}{2}$$

(Compare with Eg1: same f, different C (same end points) $\Rightarrow$ different answers)

Related integral

(2) $\int_C \vec{F}\cdot\vec{T}\,ds$

$\vec{F}: D \longmapsto \mathbb{R}^3$

$(x,y,z) \quad (F_1(x,y,z), F_2(\cdot\cdot), F_3(\cdot\cdot))$

C: a smooth curve in D

with prescribed orientation(指向)

$\vec{r}(t)$: parametrization of C with

"direction of increasing t"

= "orientation of C"

$\vec{T} = \dot{\vec{r}}(t) / |\dot{\vec{r}}(t)| \Rightarrow \vec{T}\,ds = \dot{\vec{r}}(t)\,dt$

Eg3 Evaluate $\int_C \vec{F}\cdot\vec{T}\,ds$

$$= \int_C \vec{F}\cdot d\vec{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz)$$

where $\vec{F} = (z, xy, -y^2)$

$C: \vec{r}(t) = (t^2, t, \sqrt{t}),\ 0 \le t \le 1$

<u>Sol.</u> $\dot{\vec{r}}(t) = \left(2t, 1, \frac{1}{2\sqrt{t}}\right)$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \int_{t=0}^{1} \vec{F}\cdot\frac{\dot{\vec{r}}(t)}{|\dot{\vec{r}}(t)|}\,|\dot{\vec{r}}(t)|\,dt$$

$$= \int_0^1 \underbrace{(\sqrt{t}, t^3, -t^2)}_{\vec{F}}\cdot\underbrace{\left(2t, 1, \frac{1}{2\sqrt{t}}\right)}_{\dot{\vec{r}}(t)}\,dt$$

$$= \int_0^1 \left(2t^{\frac{3}{2}} + t^3 - \frac{t^{\frac{3}{2}}}{2}\right)dt$$

$$= \frac{3}{2}\cdot\frac{2}{5} + \frac{1}{4} = \frac{17}{20}$$

Rm If C is a simple (does not intersect itself) closed curve, we use

$$\oint_C \vec{F}\cdot\vec{T}\,ds$$

to specify the orientation of C.

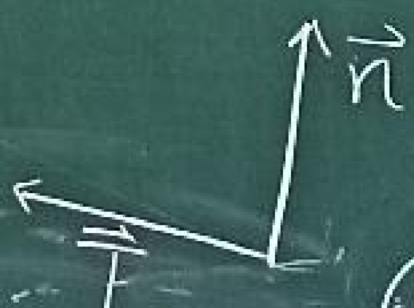
Related integral

(3) $\int_C \vec{F}\cdot\vec{n}\,ds$ (flux)

C: simple closed curve in a plane

$\vec{n}$: outward unit normal

To compute the flux,
we need to compute $\vec{n}$
from $\vec{r}(t)$. For example,
if $\vec{r}(t)$ is counterclockwise
along C, $\vec{T} = \dot{\vec{r}}(t)/|\dot{\vec{r}}(t)| = \dfrac{(\dot{x}(t), \dot{y}(t))}{\sqrt{\dot{x}^2 + \dot{y}^2}}$



$\Rightarrow$ $n_1 = T_2$

$n_2 = -T_1$

$$\therefore \vec{n}\,ds = \frac{(\dot{y}(t), -\dot{x}(t))}{\sqrt{\dot{y}^2 + \dot{x}^2}} \sqrt{\dot{x}^2 + \dot{y}^2}\,dt = (\dot{y}(t), -\dot{x}(t))\,dt$$

Eg4: $\vec{F}=(x-y,y)$, $C=\{x^2+y^2=1\}$

$$\oint_C \vec{F}\cdot\vec{T}\,ds=? \qquad \oint_C \vec{F}\cdot\vec{n}\,ds=?$$

$(\vec{F}\cdot d\vec{r})$

Sol: $C: x(t)=\cos t,\ y(t)=\sin t$, $0\le t\le 2\pi$

$\dot{x}(t)=-\sin t,\ \dot{y}(t)=\cos t$

(i) $$\oint_C \vec{F}\cdot d\vec{r}=\int_0^{2\pi}\underbrace{(F_1\cdot\dot{x}+F_2\cdot\dot{y})\,dt}_{F_1dx+F_2dy}$$

$$=\int_0^{2\pi}\big((\cos t-\sin t)(-\sin t)+\sin t\cos t\big)dt$$

$$=\int_0^{2\pi}\sin^2 t\,dt=\pi$$

(ii) $\vec{T} = (-\sin t, \cos t)$

$\vec{n} = (\cos t, \sin t)$

$$d\vec{r} = \vec{T}\,ds = (dx, dy)$$

$$\vec{n}\,ds = (dy, -dx)$$

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \int_0^{2\pi} F_1\,dy - F_2\,dx$$

$$= \int_0^{2\pi} \underline{(F_1\dot{y} - F_2\dot{x})\,dt}$$

$$= \int_0^{2\pi} (\cos t - \sin t)\cos t - \sin t(-\sin t)\,dt$$

$$= \int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt = 2\pi$$

Fundamental Theorem of line integral

<u>Thm 1</u> If f is differentiable in D, ∇f is continuous in D and C is <u>any</u> curve in D <u>from</u> point A <u>to</u> point B, then

$$\int_C \nabla f \cdot \vec{T}\, ds = f(B) - f(A)$$

Pf Let $C = \{\vec{r}(t),\ 0 \le t \le 1\}$ with $A = \vec{r}(0)$, $B = \vec{r}(1)$

$$\Rightarrow \int_C \nabla f \cdot \vec{T}\, ds = \int_0^1 \begin{pmatrix} f_x(x(t), y(t), z(t))\, \dot{x}(t) \\ + f_y(x(t), y(t), z(t))\, \dot{y}(t) \\ + f_z(x(t), y(t), z(t))\, \dot{z}(t) \end{pmatrix} dt$$

$$= \int_0^1 \frac{d}{dt} f(x(t), y(t), z(t))\, dt$$

$$= f(x(t), y(t), z(t))\Big|_{t=0}^{1} = f(B) - f(A)$$

Eg 5: Evaluate $\int_A^B \vec{F}\cdot\vec{T}\,ds$ along C, where $A=(2,0)$, $B=(-2,0)$, $C=\left\{\frac{x^2}{4}+y^2=1,\ y\geq 0\right\}$

$$\vec{F}=\frac{(x,y)}{\sqrt{x^2+y^2}}$$

Sol: $\vec{F}=\nabla\left(\sqrt{x^2+y^2}\right)$

$$\Rightarrow \int_A^B \vec{F}\cdot\vec{T}\,ds$$

$$=\int_A^B \nabla\left(\sqrt{x^2+y^2}\right)\cdot\vec{T}\,ds$$

$$=\sqrt{x^2+y^2}\,\Big|_{\underset{A}{(2,0)}}^{(-2,0)=B}=0$$

Remark: If $\vec{F} = \nabla f$

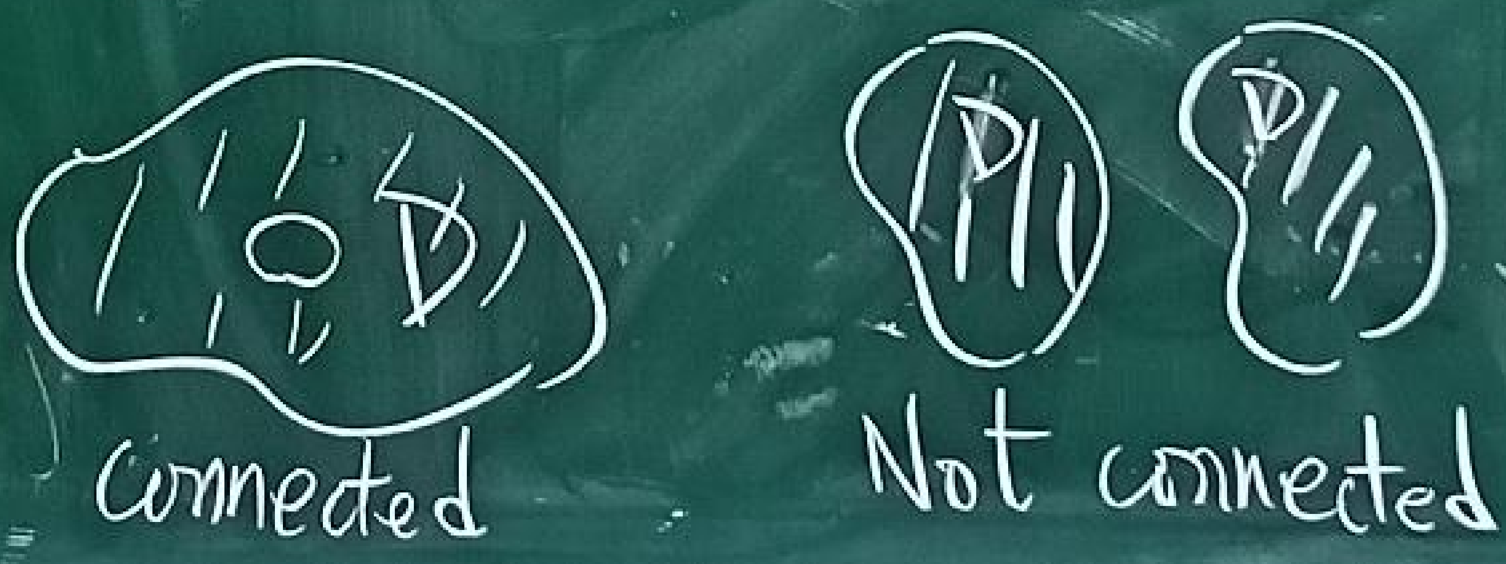
Then $\int_C \vec{F} \cdot \vec{T}\, ds = 0$

for any closed curve C

Def: D is "connected"

If any two point in D

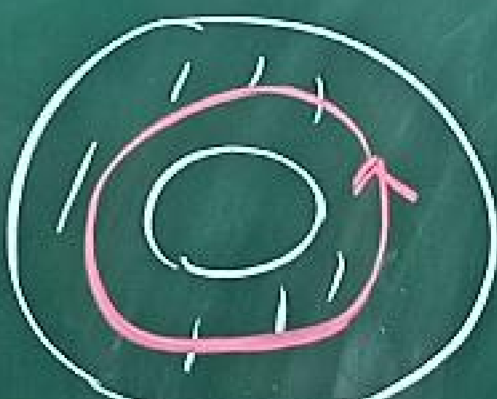
can be connected by a path (curve) in D



Def D is simply connected

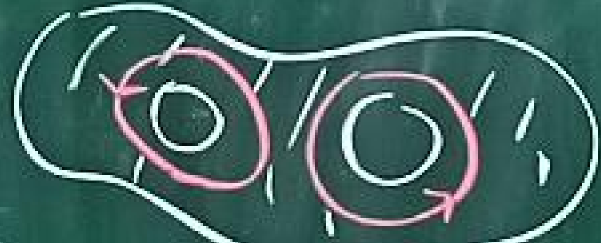
If any closed curve in D
can be continuously shrinked
to a point without leaving D
in D

Eg



$D = \{(x,y),\ 1 \le x^2+y^2 \le 4\}$

Not simply connected



: No

$\boxed{\begin{array}{c} D = \mathbb{R}^2 \setminus \{(0,0)\} \\ \underline{\text{No}} \end{array}}$

$D = \{(x,y,z)\ 1 \le x^2+y^2 \le 4\}$: No

$D = \{(x,y,z),\ 1 \le x^2+y^2+z^2 \le 4\}$, Yes

$D = \mathbb{R}^3 \setminus \{(0,0,0)\}$: Yes