

Substitution in Multiple Integrals

Eg 1: ~~$\frac{y}{x}=z$~~ Find area of $R = \left\{ \begin{array}{l} 1 \leq xy \leq 2 \\ 1 \leq \frac{y}{x} \leq 2 \end{array} \right\}$

~~$\frac{y}{x}=1$~~

R

$xy=2$

$xy=1$

$= \left\{ \begin{array}{l} 1 \leq u \leq 2 \\ 1 \leq v \leq 2 \end{array} \right\}$

$$A = \int_{v=1}^2 \int_{u=1}^2 dA, \quad \underline{Q:} \quad dA = ? \, du \, dv$$

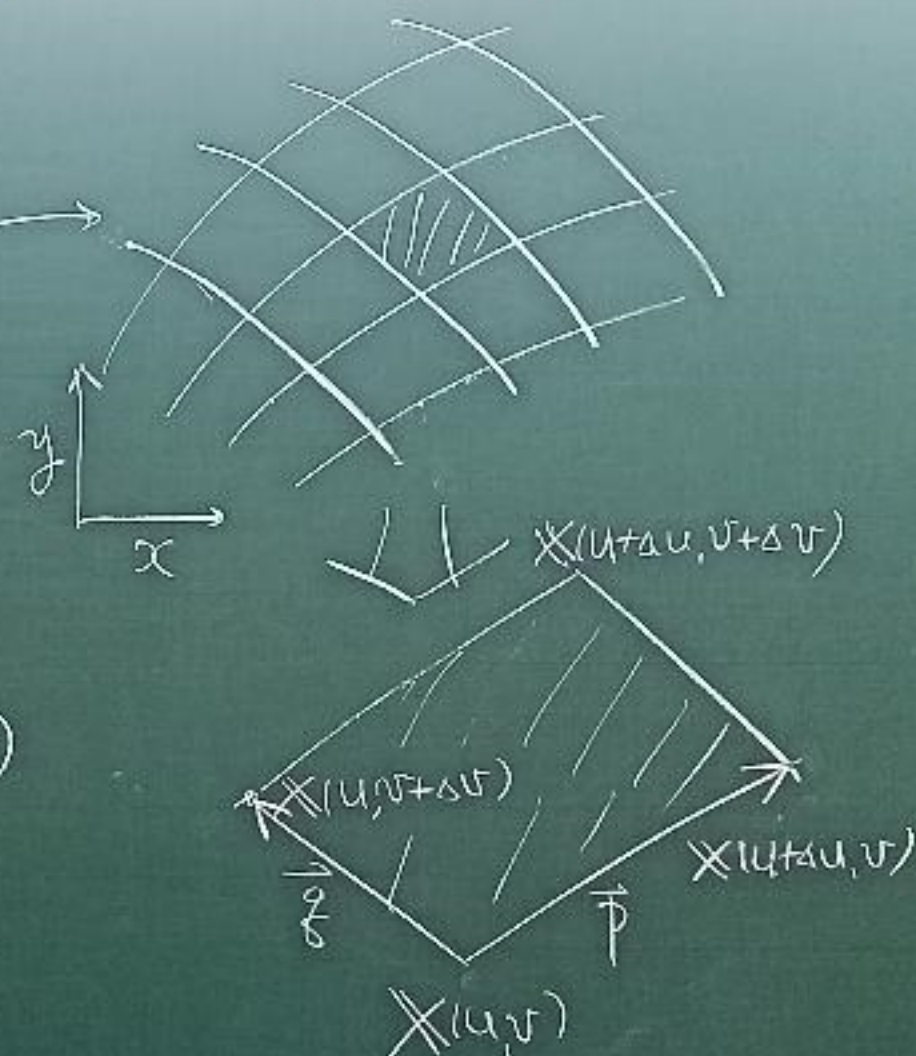
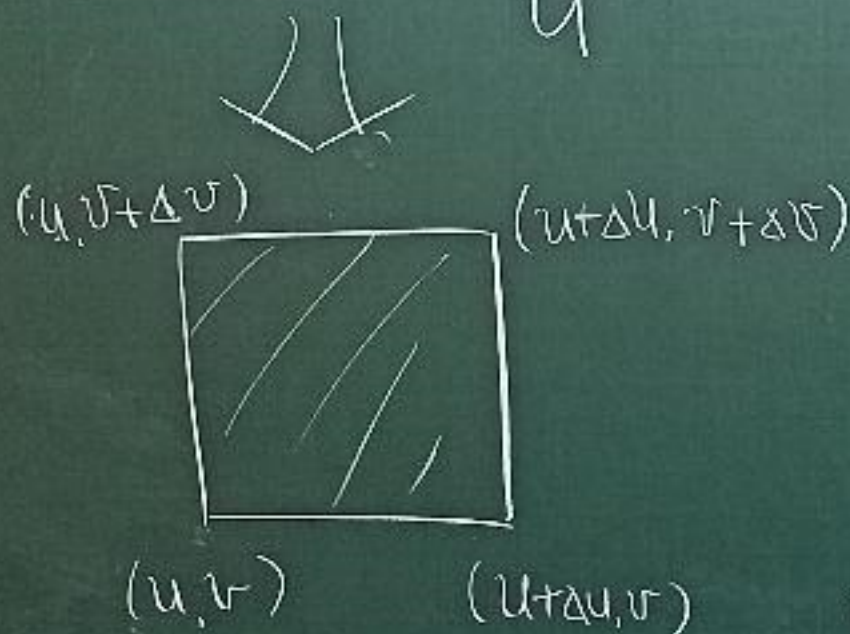
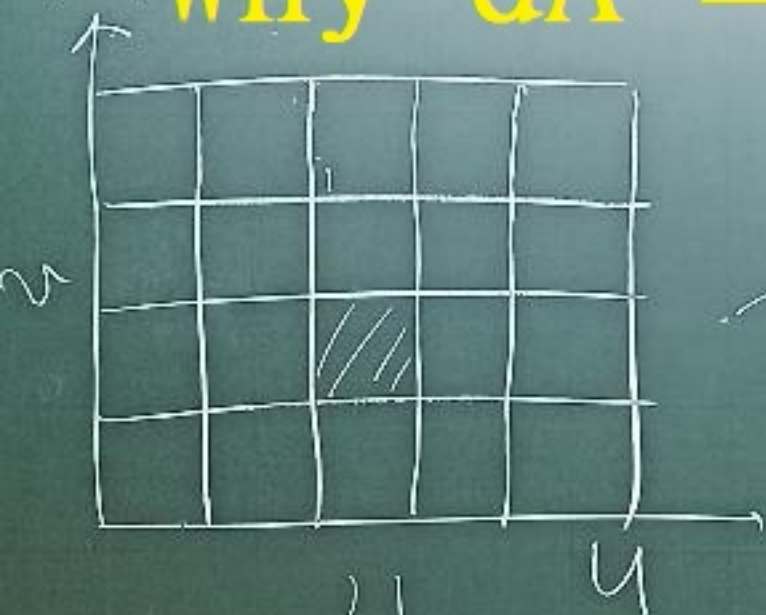
Ans: $dA = \left| \det \left(\frac{\partial(x,y)}{\partial(u,v)} \right) \right| du \, dv$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Need $\begin{cases} u=xy \\ v=\frac{y}{x} \end{cases}$

$$\rightarrow \begin{cases} x = x(u,v) = \sqrt{\frac{u}{v}} \\ y = y(u,v) = \sqrt{uv} \end{cases}$$

Why $dA = |J| du dv$:



$$\mathbf{X}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ \vdots \end{pmatrix} \text{ etc.}$$

$$\mathbf{X}(u+\Delta u, v) = \begin{pmatrix} x(u+\Delta u, v) \\ y(u+\Delta u, v) \\ \vdots \end{pmatrix}$$

$$\Delta A \approx |\vec{p} \times \vec{z}| \quad \text{See page 4}$$

for (a), (b)

$$= |(\cancel{X(u+\Delta u, v)} - \cancel{X(u, v)}) \times (\cancel{X(u, v+\Delta v)} - \cancel{X(u, v)})|$$

$$\downarrow \quad \left| \underbrace{\frac{\partial \cancel{X}}{\partial u}(u+\cancel{C}\Delta u, v)}_{(a)} \Delta u \times \underbrace{\frac{\partial \cancel{X}}{\partial v}(u, v+\cancel{C}\Delta v)}_{(b)} \Delta v \right|$$

Mean Value Theorem

$$= \left| \det \begin{pmatrix} \frac{\partial \cancel{X}}{\partial u} & \frac{\partial \cancel{X}}{\partial v} \end{pmatrix} \right| \Delta u \Delta v \left(\begin{array}{l} \text{Note} \\ \underbrace{\cancel{X}_u}_{(a)} = \begin{pmatrix} x_u \\ y_u \end{pmatrix} \\ \underbrace{\cancel{X}_v}_{(b)} = \begin{pmatrix} x_v \\ y_v \end{pmatrix} \end{array} \right)$$

Note: $(a, b, 0) \times (c, d, 0)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & 0 \\ c & d & 0 \end{vmatrix} \left(\begin{array}{l} \therefore \text{we write} \\ (a, b) \times (c, d) \\ = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array} \right)$$

$$= (0, 0, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix})$$

Note: (a) and (b) on page 3 should be

$$\begin{pmatrix} \frac{\partial x}{\partial u}(u + c_{11}\Delta u, v) \\ \frac{\partial y}{\partial u}(u + c_{12}\Delta u, v) \end{pmatrix} \text{ and } \begin{pmatrix} \frac{\partial x}{\partial v}(u, v + c_{21}\Delta v) \\ \frac{\partial y}{\partial v}(u, v + c_{22}\Delta v) \end{pmatrix}$$

respectively. Here c_{11} and c_{12} may be different.

Similarly for X_u , X_v , X_w on page 6.

$$\therefore \Delta A = |J| \Delta u \Delta v$$

$$J = \det \left(\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v} \right)$$

$$= \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Back to Eg 1:

$$A = \int_{u=1}^2 \int_{v=1}^2 dA \quad \begin{array}{l} u = xy \\ v = \frac{y}{x} \end{array}$$

$$= \int_1^2 \int_1^2 |J| du dv \quad \begin{array}{l} x = \sqrt{\frac{u}{v}} \\ y = \sqrt{uv} \end{array}$$

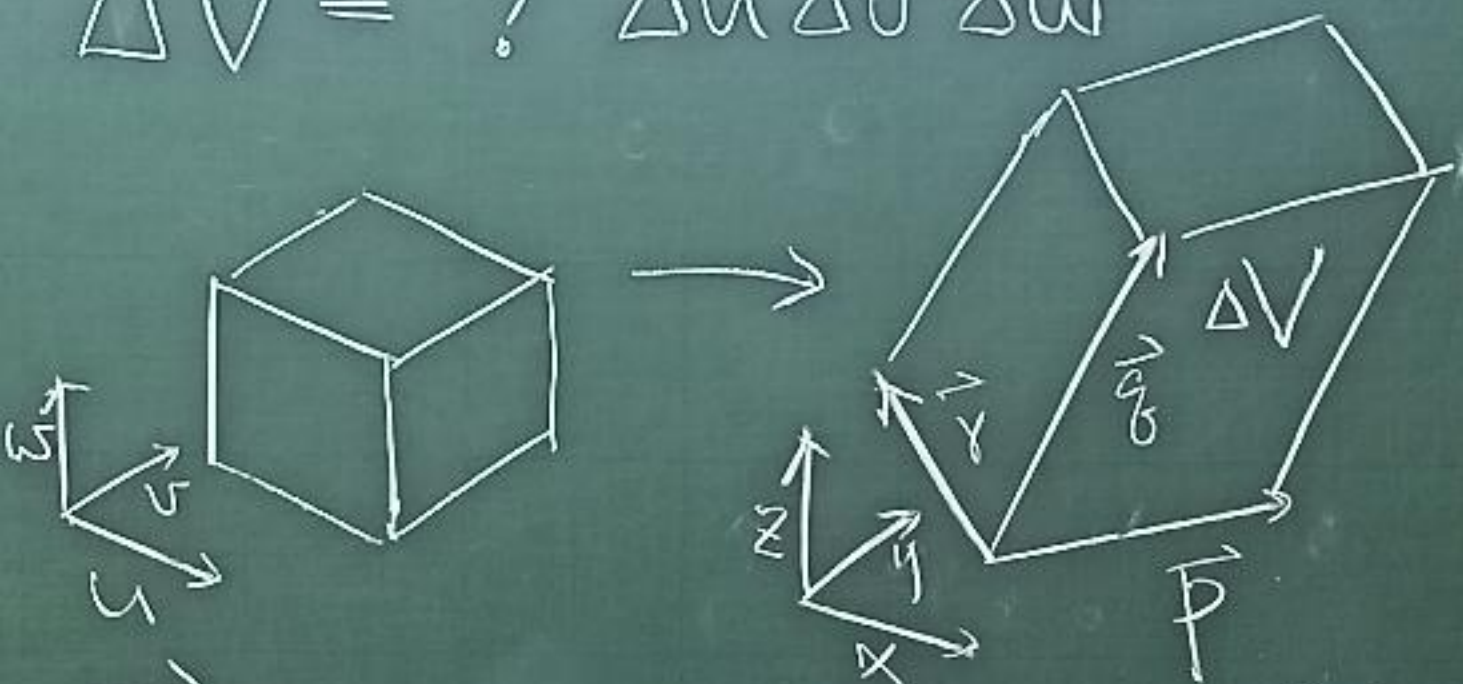
$$= \int_1^2 \int_1^2 \left| \det \begin{pmatrix} \frac{1}{2} \frac{1}{\sqrt{uv}} & -\frac{1}{2} \sqrt{\frac{u}{v^3}} \\ \frac{1}{2} \sqrt{\frac{v}{u}} & \frac{1}{2} \sqrt{\frac{u}{v}} \end{pmatrix} \right| du dv$$

$$= \int_1^2 \int_1^2 \frac{1}{2v} du dv = \ln 2 / 2$$

For Triple Integrals

$$U(x, y, z), V(x, y, z), W(x, y, z)$$

$$\Delta V = ? \Delta U \Delta V \Delta W$$



$$\vec{p} = X(u+\Delta u, v, w) - X(u, v, w) \approx X_u \Delta u$$

$$\vec{q} = X(u, v+\Delta v, w) - X(u, v, w) \approx X_v \Delta v$$

$$\vec{r} = X(u, v, w+\Delta w) - X(u, v, w) \approx X_w \Delta w$$

$$\Delta V = |\vec{p} \times \vec{q} \cdot \vec{r}| = \left| \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} \right| \Delta u \Delta v \Delta w$$
$$= |J| \Delta u \Delta v \Delta w$$

Summary

Step 1 Find suitable

$u(x, y, z), v(x, y, z), w(x, y, z)$
for the problem.

Step 2 Solve for $x(u, v, w)$
 $y(u, v, w), z(u, v, w)$

Step 3. $J = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$

$$\Delta V = |J| \Delta u \Delta v \Delta w$$

Eg. 2:
$$\int_{z=0}^3 \int_{y=0}^4 \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \left(\frac{2x-y}{z} + \frac{z}{3} \right) dx dy dz$$
 Evaluate

Using $u = x - \frac{y}{2}$, $v = \frac{y}{2}$, $w = \frac{z}{3}$

Sol $x = u + v$, $y = 2v$, $z = 3w$

$dw: \int_{z=0}^3 = \int_{w=0}^1, dv: \int_{y=0}^4 = \int_{v=0}^2$

$du: \int_{x=\frac{y}{2}}^{x=\frac{y}{2}+1} \quad x = \frac{y}{2} \Leftrightarrow u = 0$
 $x = \frac{y}{2} + 1 \Leftrightarrow u = 1$

$J = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 6$

$$I = \int_{w=0}^1 \int_{v=0}^2 \int_{u=0}^1 (u+w) \underline{6 \, du \, dv \, dw}$$

$$= 6 \left(\iiint u \, du \, dv \, dw + \iiint w \, du \, dv \, dw \right)$$

$$= 6 \left(\int_{w=0}^1 \int_{v=0}^2 \left. \frac{u^2}{2} \right|_0^1 \, dv \, dw + \int_{v=0}^2 \int_{u=0}^1 \left. \left(\frac{w^2}{2} \right) \right|_0^1 \, du \, dv \right)$$

$$= 6 \left(\frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \cdot 1 \cdot 2 \right)$$

$$= 12$$

Eg 3: Spherical Coordinate

$$x = r \cos \theta = \rho \sin \phi \cos \theta$$

$$y = r \sin \theta = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$J = \det \begin{pmatrix} x_\theta & x_\phi & x_\rho \\ y_\theta & y_\phi & y_\rho \\ z_\theta & z_\phi & z_\rho \end{pmatrix}$$

$$= \det \begin{pmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{pmatrix} = \rho^2 \sin \phi \quad (\text{check!})$$

Line integrals

Goal $\int_{\vec{p}}^{\vec{q}} \nabla f(\vec{r}) \cdot \underbrace{\vec{z}}_{d\vec{r}} ds = f(\vec{q}) - f(\vec{p})$

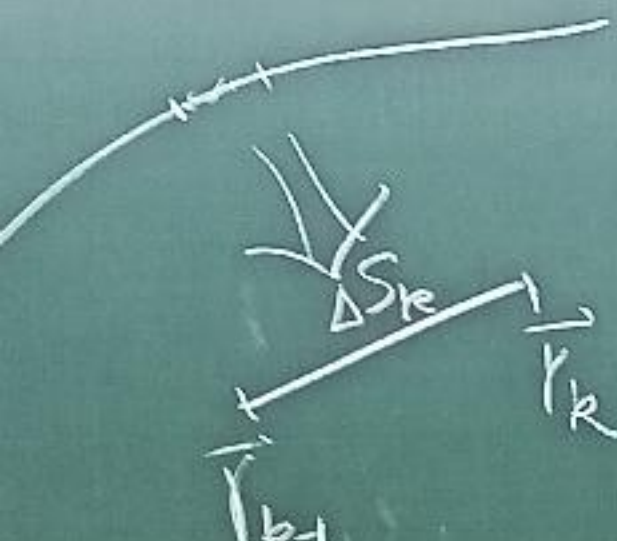
$(\vec{p}, \vec{q} \in \mathbb{R}^3 \quad f: \mathbb{R}^3 \xrightarrow{d\vec{r}} \mathbb{R})$

We start with a related integral

$$\int_C f(x, y, z) ds = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k, y_k, z_k) \Delta S_k$$

$D \subseteq \mathbb{R}^3$: domain of definition of f

$C \subseteq D$: a smooth curve.



$$C = \{ \vec{r}(t), a \leq t \leq b \}$$

$$= \left\{ \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, a \leq t \leq b \right\}$$

$$\vec{r}_k = \vec{r}(t_k)$$

$$\Delta t_k = t_k - t_{k-1}$$

$$\Delta S_k = |\vec{r}_k - \vec{r}_{k-1}|$$

$$= \frac{|\vec{r}_k - \vec{r}_{k-1}|}{\Delta t_k} \Delta t_k$$

$$\therefore dS = \left| \frac{d\vec{r}(t)}{dt} \right| dt$$

Line integrals

Eg 1. Evaluate $\int_C f(x, y, z) ds$

where $f = x - 3y^2 + z$

C = line segment between
 $(0, 0, 0)$ and $(1, 1, 1)$.

Sol: Step 1: Find $\vec{r}(t)$ for C

such as $\vec{r}(t) = (t, t, t)$, $0 \leq t \leq 1$

Step 2: $ds = |\vec{r}'(t)| dt = |(1, 1, 1)| dt$

Step 3: $\text{Ans} = \int_{t=0}^1 (t - 3t^2 + t) \sqrt{3} dt = 0$

Rm $ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$

Rm: the value of $I = \int_C f(x, y, z) ds$,
is independent of the parameter
 $\vec{r}(t)$. For example

$$\vec{r}(t) = (t^2, t^2, t^2), \quad 0 \leq t \leq 1$$

$$\text{or } \vec{r}(t) = (1-t, 1-t, 1-t), \quad 0 \leq t \leq 1$$

all give the same value of

I as long as $\vec{r}(t)$ is
a correct parametrization
for C .