Triple Integrals in Cylindrical and Spherical Coordinates

I. Cylindrical Coordinate

= polar coordinate + z coord.

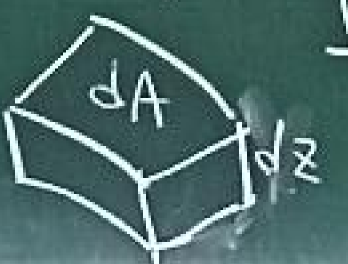
$$(x, y, z) \longleftrightarrow (r, \theta, z)$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$r = \sqrt{x^2+y^2}, \quad \tan\theta = \frac{y}{x}, \quad z = z$$

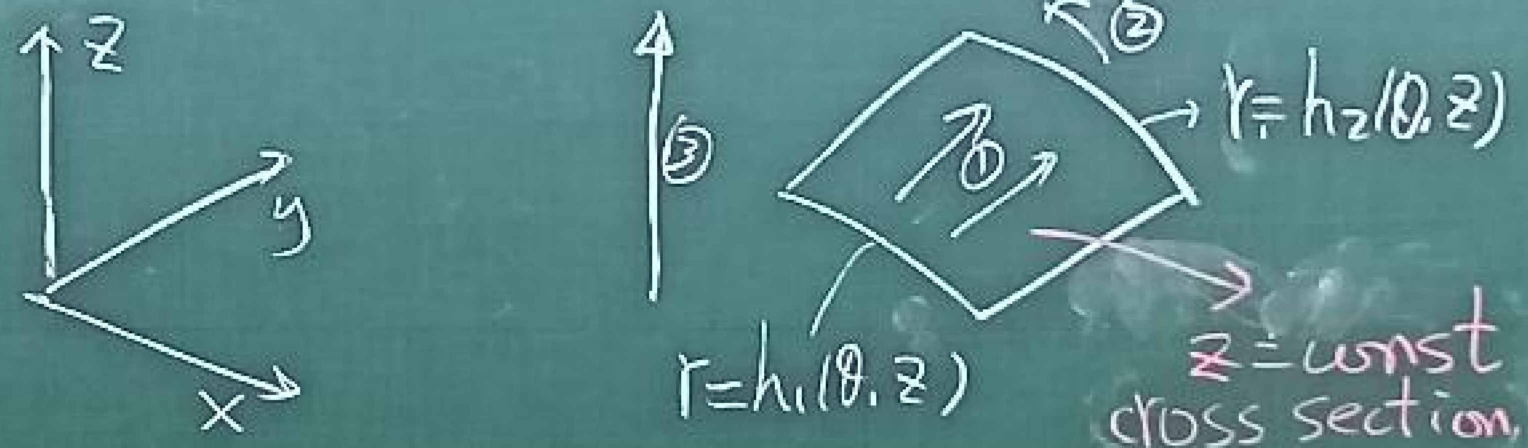
$$\theta = \begin{cases} \tan^{-1}\frac{y}{x} \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), & (x,y) \in \mathrm{I}, \mathrm{IV} \\ \tan^{-1}\frac{y}{x} \pm \pi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \left(\frac{-3\pi}{2}, \frac{\pi}{2}\right) & (x,y) \in \mathrm{II}, \mathrm{III} \end{cases}$$

$$dV = dA \cdot dz = r\,dr\,d\theta\,dz$$

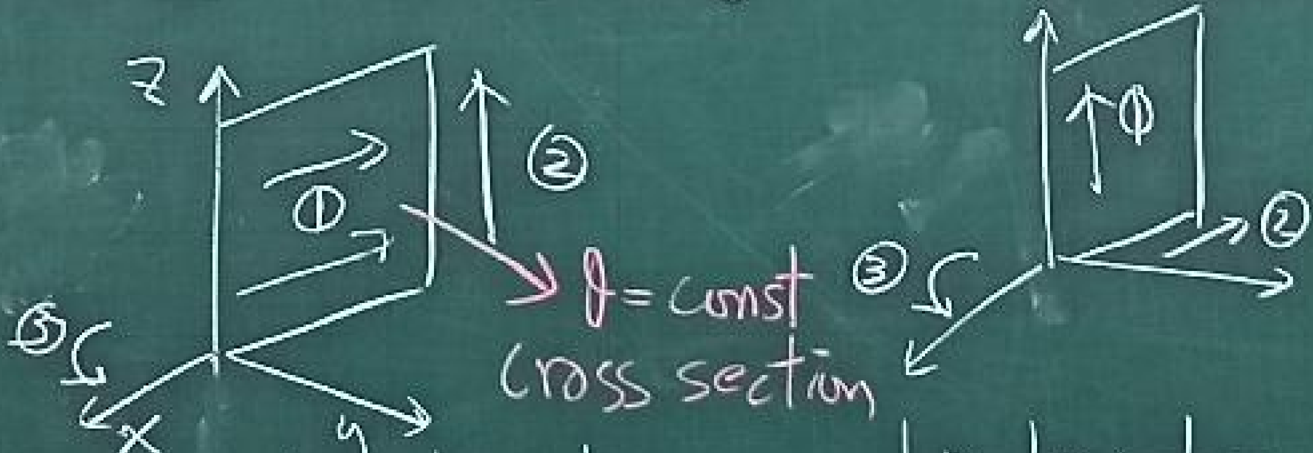


Finding limits of integration

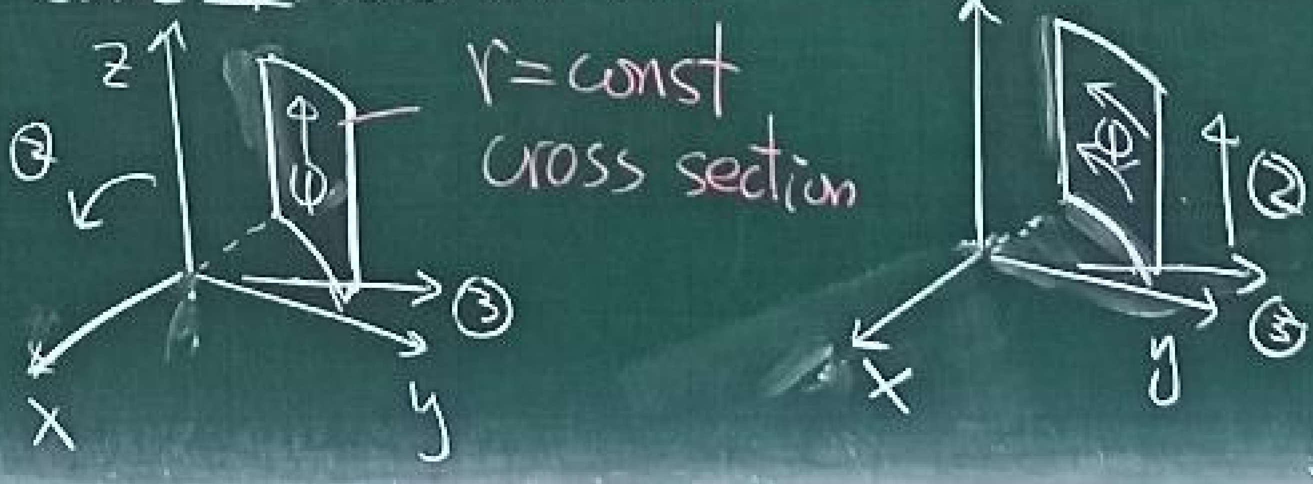
Case I: $r\,dr\,d\theta\,dz$ ($d\theta\,r\,dr\,dz$ is similar)



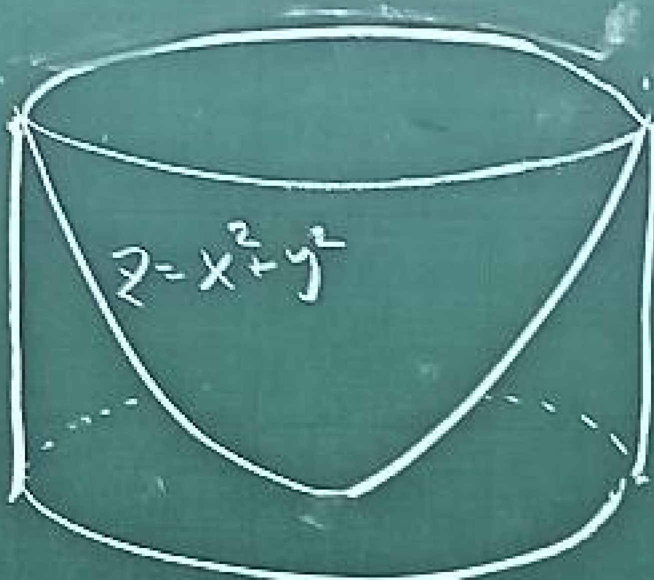
Case II $r\,dr\,dz\,d\theta$ or $dz\,r\,dr\,d\theta$



Case III $dz\,d\theta\,r\,dr$ or $d\theta\,dz\,r\,dr$



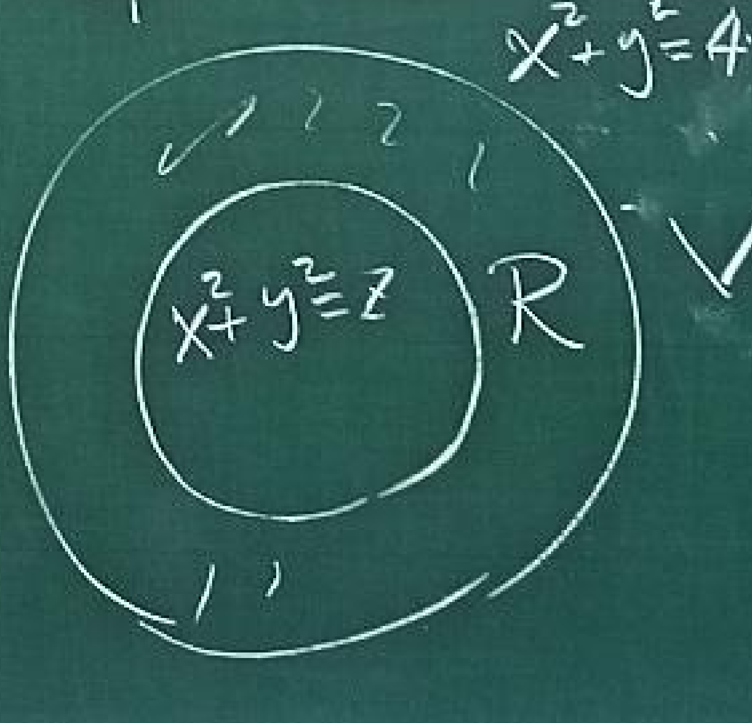
Eg $D=\left\{\begin{matrix} x^2+y^2\le 4 \\ 0\le z\le x^2+y^2\end{matrix}\right\}$



Find volume of D using Cylindrical coordinates.

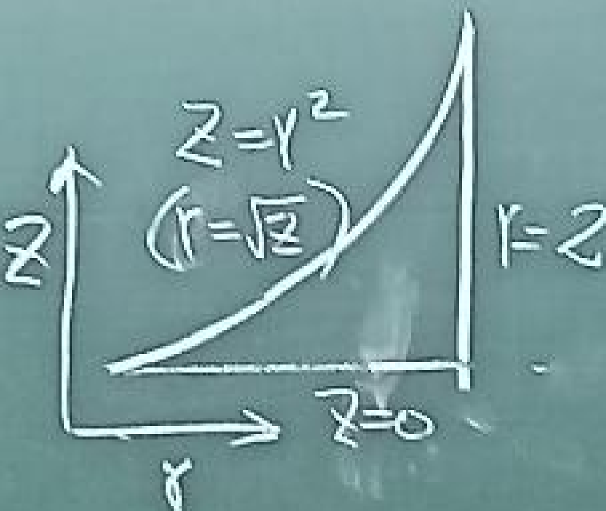
case I: $r\,dr\,d\theta\,dz$

Step 1 cross section



$$V=\int_0^4\int_0^{2\pi}\int_{r=\sqrt{z}}^{2} r\,dr\,d\theta\,dz$$

Case II $\quad r\,dr\,dz\,d\theta$ or $dz\,r\,dr\,d\theta$

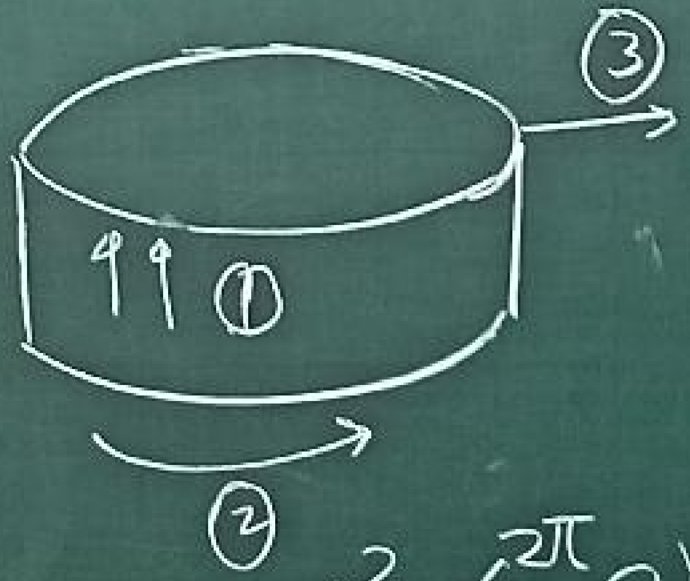


$$V=\int_0^{2\pi}\int_0^{4}\int_{r=\sqrt{z}}^{2} r\,dr\,dz\,d\theta$$

$$=\int_0^{2\pi}\int_0^{2}\int_{z=0}^{r^2} dz\,r\,dr\,d\theta$$

$$=8\pi$$

Case III $\quad dz\,d\theta\,r\,dr$

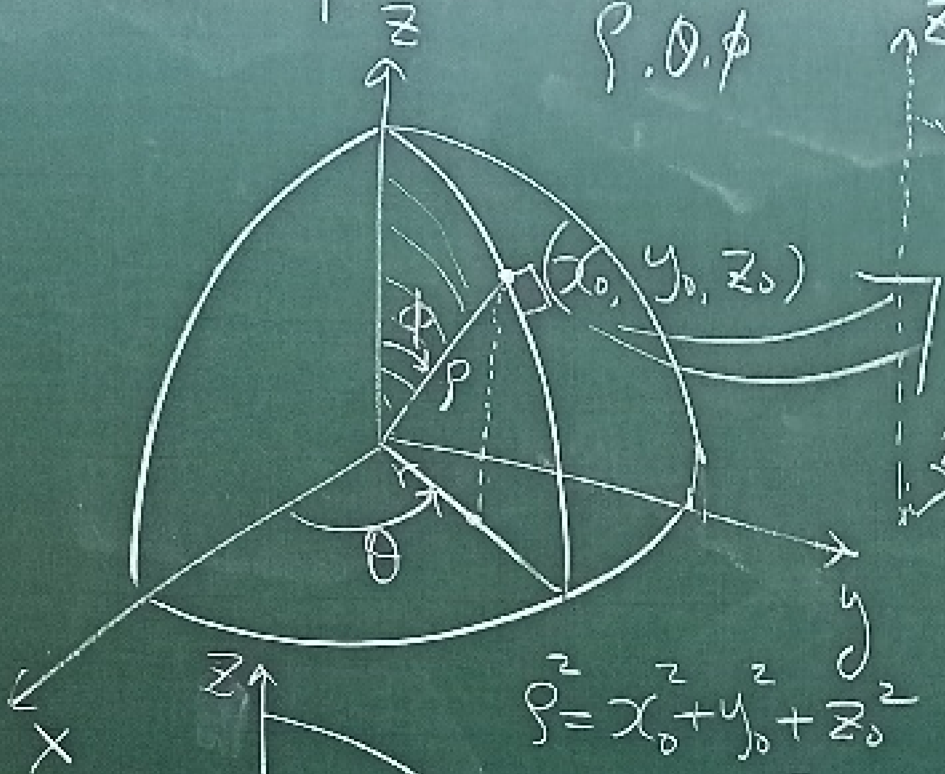


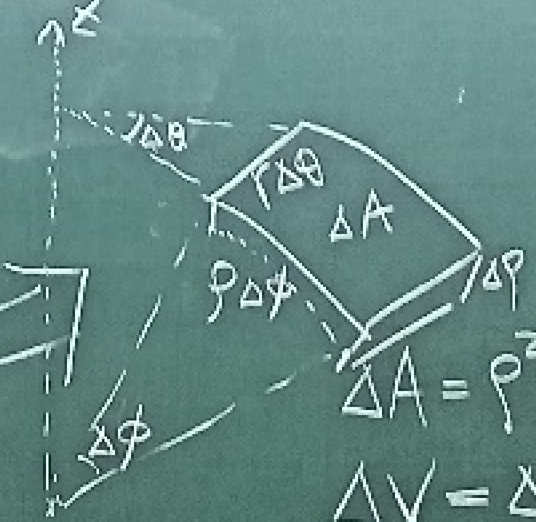
$$V=\int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{r^2} dz\,d\theta\,r\,dr$$

$$=8\pi$$

II Spherical Coordinates

$\rho . \theta . \phi$

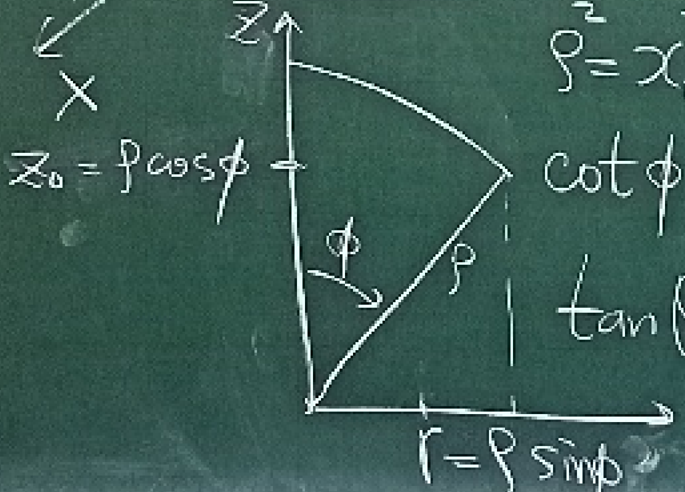




$$\Delta A = \rho^2 \sin\phi \, \Delta\theta \, \Delta\phi$$

$$\Delta V = \Delta A \, \Delta\rho$$

$$= \rho^2 \sin\phi \, \Delta\rho \, \Delta\theta \, \Delta\phi$$



$$\rho^2 = x_0^2 + y_0^2 + z_0^2$$

$$\cot\phi = \frac{z_0}{\sqrt{x_0^2 + y_0^2}} = \frac{z_0}{r}$$

$$\tan\theta = \frac{y_0}{x_0}$$

$$x_0 = r\cos\theta$$

$$= \rho \sin\phi \cos\theta$$

$$y_0 = r\sin\theta$$

$$= \rho \sin\phi \sin\theta$$

$$z_0 = \rho\cos\phi$$

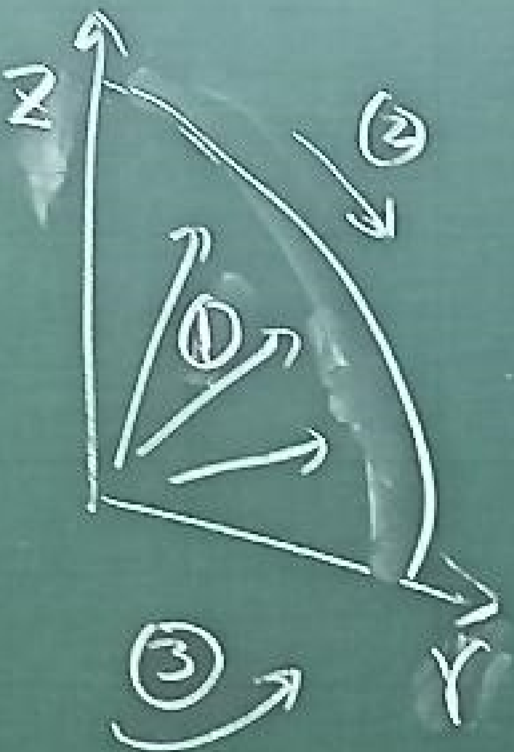
Cross Sections for various order of integration

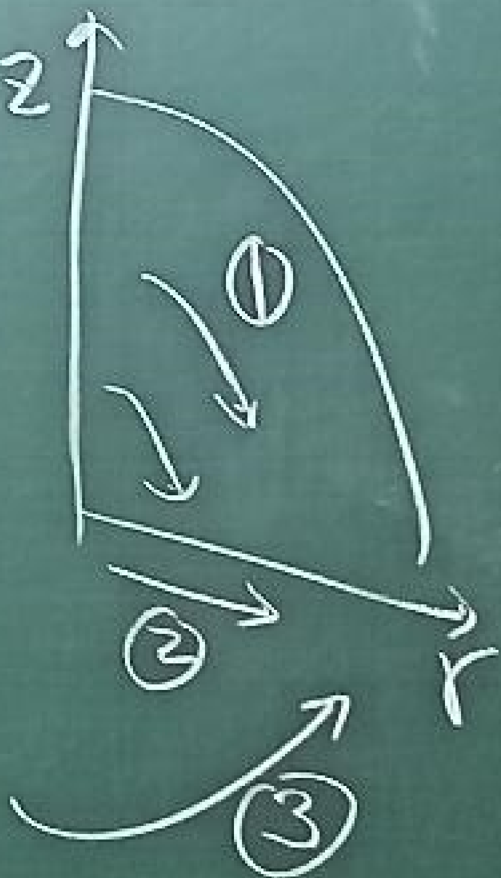
Case I: $d\rho\, d\theta\, d\phi$ or $d\theta\, d\rho\, d\phi$

$\phi = \text{constant}$

Case II $d\theta\, d\phi\, d\rho$ $\quad$ $d\phi\, d\theta\, d\rho$

Case III $d\rho\, d\phi\, d\theta$, $d\phi\, d\rho\, d\theta$



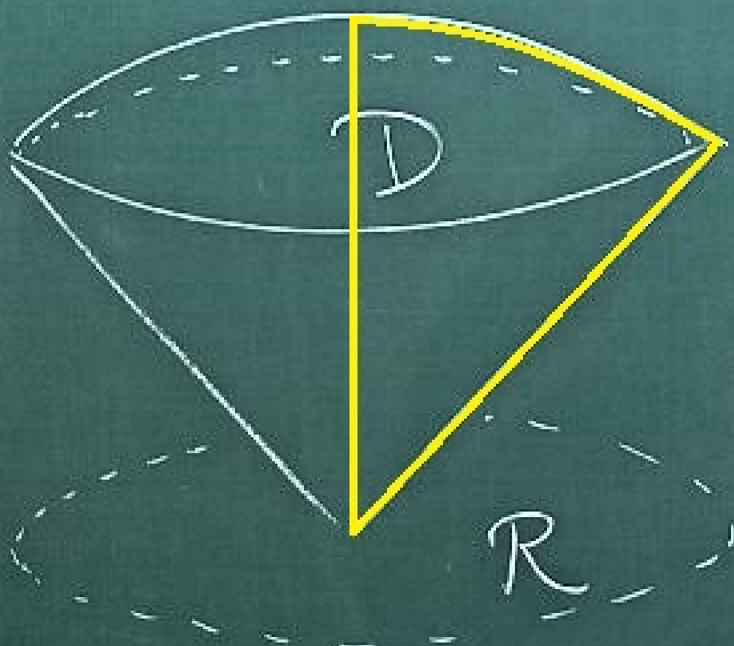


Easiest case in general

Must learn!

Example. $D = \left\{ \begin{array}{l} x^2+y^2+z^2 \le 1 \\ z \ge \sqrt{\dfrac{x^2+y^2}{3}} \end{array} \right\}$

Find volume of D



Sol (I). Cylindrical coordinate

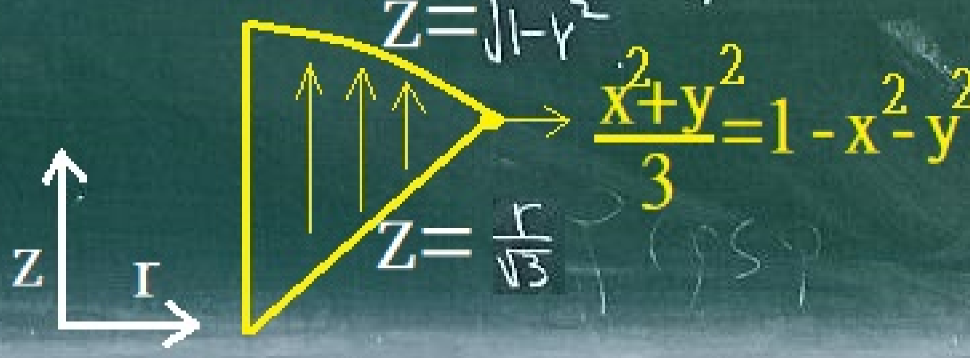
$$R = \left\{ \begin{array}{l} (x,y)\ \exists\, z \in \mathbb{R} \\ \sqrt{\dfrac{x^2+y^2}{3}} \le z \le \sqrt{1-(x^2+y^2)} \end{array} \right\}$$

$$= \left\{ \frac{x^2+y^2}{3} \le 1-(x^2+y^2) \right\}$$

$$= \left\{ x^2+y^2 \le \frac{3}{4} \right\}$$

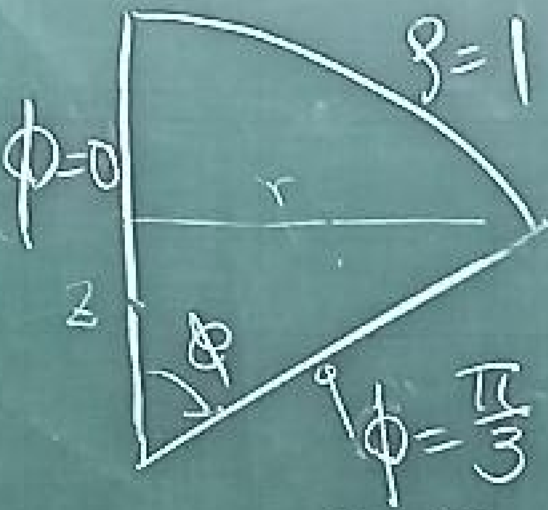
$$V = \iint_R \int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} dz\, dA$$

$$= \int_0^{2\pi} \int_{r=0}^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{r}{3} \right) r\, dr\, d\theta$$



$\underline{\text{II}}$ Spherical Coordinate

$(x^2+y^2+z^2=1)$



$$\left(\frac{r}{z}=\sqrt{3},\ \tan\phi=\sqrt{3}\right)$$

$$z=\sqrt{\frac{x^2+y^2}{3}}$$

$$D=\left\{\begin{array}{l} 0\le\rho\le 1 \\ 0\le\phi\le\frac{\pi}{3} \\ 0\le\theta\le 2\pi \end{array}\right\}$$

$$V=\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{1}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$=\underbrace{\frac{1}{3}}_{d\rho}\underbrace{\left(1-\frac{1}{2}\right)}_{d\phi}\underbrace{2\pi}_{d\theta}=\frac{\pi}{3}$$