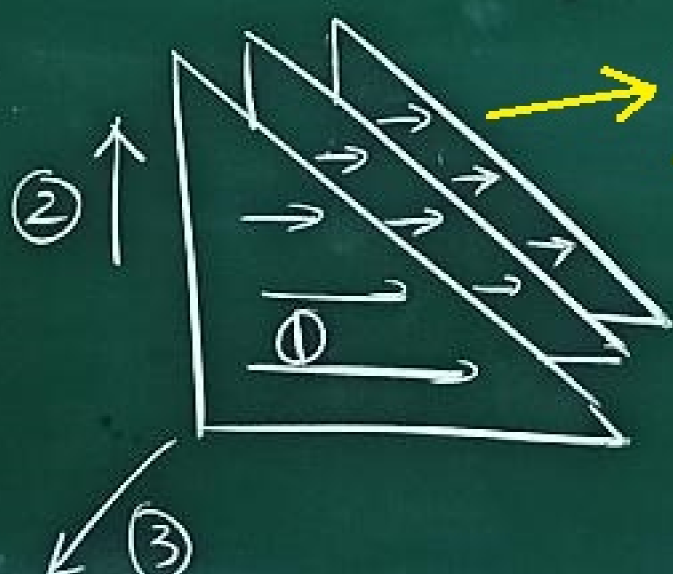
Triple integrals in Cartesian Coordinates

$$I = \iiint_D f(x,y,z)\,dV = \lim_{\|P\| \to 0} \sum_{k=0}^{n} f(x_k, y_k, z_k)\,\Delta V_k$$

When $f(x,y,z) \equiv 1$, $I =$ volume of D

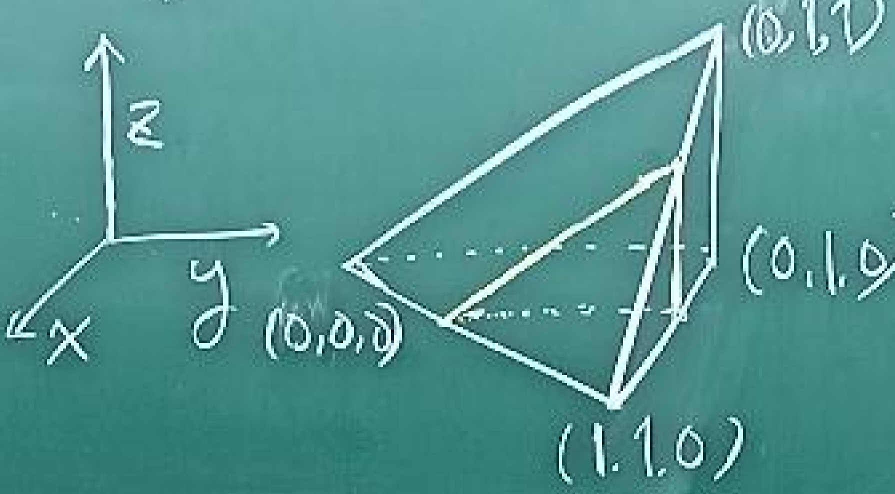
In Cartesian Coordinates

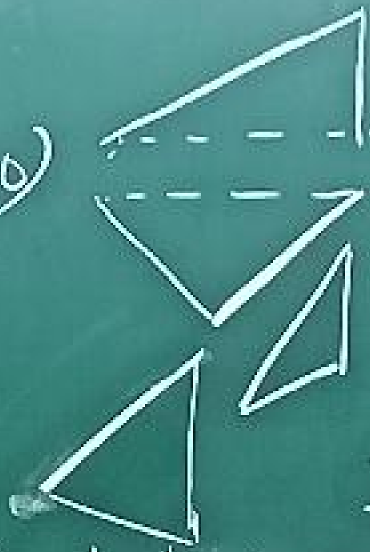
$$dV = dx\,dy\,dz = dy\,dz\,dx = \cdots\cdots$$



③ = constant

Eg 1 Find the volume of the domain enclosed by





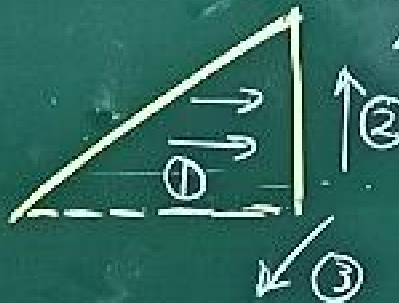
$x=0$ (A)

$z=0$ (B)

$y=1$ (C)

$x-y+z=0$ (D)

Method 1: $\overset{①}{dy}\,\overset{②}{dz}\,\overset{③}{dx}$ (cut along ③: $x=$ constant)



$$I=\int_{x=0}^{1}\int_{z=0\,(B)}^{z=1-x\,(C\cap D)}\int_{y=x+z\,(D)}^{y=1\,(C)} 1\,dy\,dz\,dx$$

① dy: need (C) and (D) in the form $y=f(x,z)$

$y=1$ $\quad$ $y=x+z$

② dz: need (B) and (C∩D) in the form $z=g(x)$

$z=0$ $\quad$ $z=1-x$ (eliminate y from (C),(D))

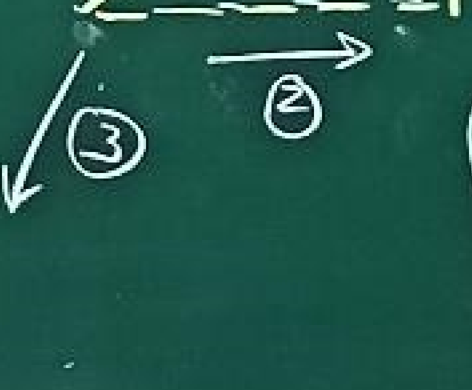
$$\therefore\ I = \int_{x=0}^{1}\int_{z=0}^{1-x}\int_{y=x-z}^{1} dy\, dz\, dx$$

$$= \int_{x=0}^{1}\int_{z=0}^{1-x} (1-x+z)\, dz\ dx$$

$$= \int_{x=0}^{1} (1-x)z + \frac{z^2}{2}\Bigg|_{z=0}^{1-x} dx$$

$$= \int_{x=0}^{1} \frac{1}{2}(1-x)^2 dx = \frac{1}{6}$$

Method 2 : $\underset{①}{dz}\ \underset{②}{dy}\ \underset{③}{dx}$



$\rightarrow$ Cut along ③ = const

Step ①:

$$\int_{(B)}^{(D)} 1\, dz = \int_{z=0}^{z=y-x} dz = y-x$$

Need (B), (D) as $z = F(x,y)$

Step ②

$$\int_{B\cap D}^{C} (y-x)\,dy = \int_{y=x}^{1} (y-x)\,dy$$

(Need $B\cap D$ and C as $y=G(x)$
by eliminating z ($=$ ①))

$$= \left[\frac{y^2}{2} - xy\right]_{y=x}^{1} dy = \frac{(1-x)^2}{2}$$

Step ③ $$\int_0^1 \frac{(1-x)^2}{2}\,dx = \frac{1}{6}$$

Other orders: $dx\,dy\,dz$,
$dy\,dz\,dx$, ...

(Homework)

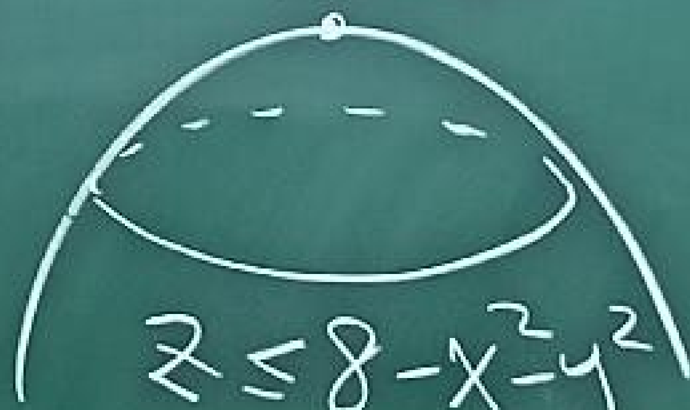
Eg2. Find the volume of

$$D = \{ x^2+3y^2 \le z \le 8-x^2-y^2 \}$$

$z \ge x^2+3y^2$

$(0,0,0)$

$(0,0,8)$

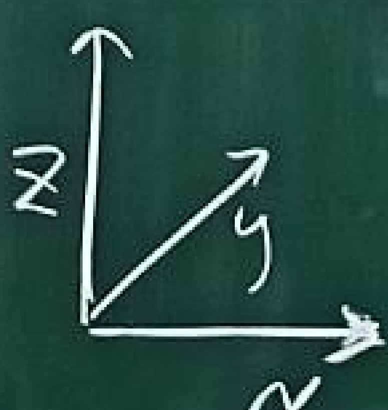


$z \le 8-x^2-y^2$

$\Downarrow$

$\textcircled{3} = ?$

Not z!



$$\begin{cases} d\underset{z}{\textcircled{1}}\,d\underset{y}{\textcircled{2}}\,dx \\ \text{or} \\ \text{or } d\underset{z}{\textcircled{1}}\,d\underset{x}{\textcircled{2}}\,dy \end{cases}$$

are preferred (easier)

For example $dz\,dy\,dx$

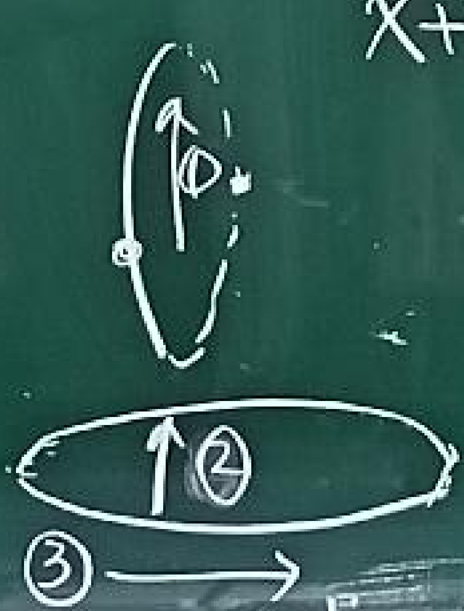
$$V=\iint_R \int_{z=x^2+3y^2}^{8-x^2-y^2} 1\,dz\,dA$$

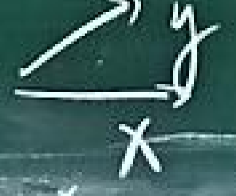
$$R=\{(x,y)\mid x^2+3y^2\le 8-x^2-y^2\}$$

(So that $\int_{z=x^2+y^2}^{8-x^2-y^2} dz$ is meaningful (positive))

$$=\{x^2+2y^2\le 4\}$$

$$V=\iint_{x^2+2y^2\le 4} 8-2x^2-4y^2\,dy\,dx$$





(<u>Rm</u>. Let $Y=2y$

$dY=2\,dy$

$$\iint_{x^2+Y^2\le 4} 8-2(x^2+Y^2)\,\frac{dY}{2}\,dx$$

Polar coordinate integration)

$$\iint dy\,dx = \int_{x=?}^{?}\int_{y=?}^{?} dy\,dx$$

$$x^2+2y^2 \le 4$$

$$\Rightarrow -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}}$$

$$R = \left\{ \begin{array}{l} -\sqrt{\frac{4-x^2}{2}} \le y \le \sqrt{\frac{4-x^2}{2}} \\ -2 \le x \le 2 \end{array} \right\}$$

$$\therefore V = \int_{-2}^{2}\int_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8-2x^2-y^2)\,dy\,dx$$

$$= \int_{-2}^{2} (8-2x^2)y - \frac{4y^3}{3} \Bigg|_{y=-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \frac{8}{3}(4-x^2)\sqrt{\frac{4-x^2}{2}}\,dx \quad \left(\begin{array}{l} x = 2\sin\theta \\ dx = 2\cos\theta\,d\theta \end{array} \right)$$

Triple Integrals in Cylindrical Coord.

Cylindrical coordinate
= polar coordinate + z coordinate

$$(x, y, z) \longleftrightarrow (r, \theta, z)$$

$x = r\cos\theta$ $\qquad$ $r = \sqrt{x^2+y^2}$

$y = r\sin\theta$ $\qquad$ $\tan\theta = \frac{y}{x}$

$z = z$ $\qquad$ $z = z$

Here $\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right), & (x,y) \in \mathrm{I}, \mathrm{IV} \\ \tan^{-1}\left(\frac{y}{x}\right) \pm \pi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) & (x,y) \in \mathrm{II}, \mathrm{III} \\ \qquad \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right) & \end{cases}$

$$dV = dA\,dz = r\,dr\,d\theta\,dz$$

(or other ordering)