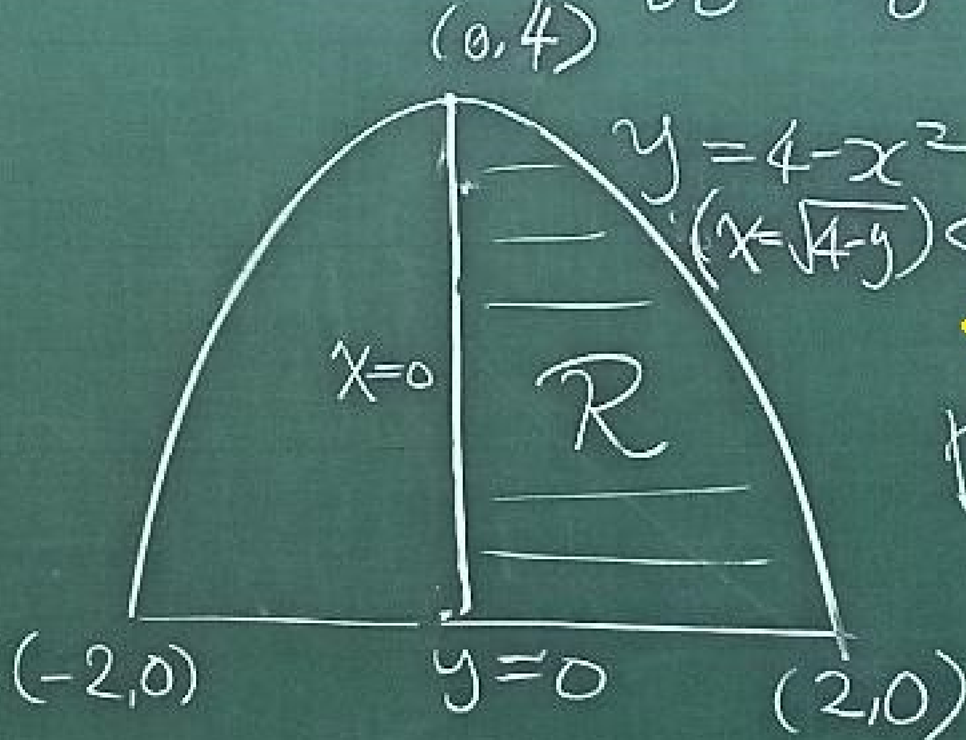
Eg4: $\displaystyle\int_0^2 \int_0^{4-x^2} \frac{x e^{2y}}{4-y}\, dy\, dx = ?$

Sol: Try $dx\,dy$ instead.

Note: $\neq \displaystyle\int_0^{4-x^2} \int_0^2 \cdots \, dx\, dy$



Step 1
plot R.

Step 2: Find limits of integration for dx

left: $x=0$

Right: $y=4-x^2$, $x=\sqrt{4-y}$

Step 3

$$A = \int_{y=0}^{4} \int_{x=0}^{\sqrt{4-y}} \frac{x e^{2y}}{4-y}\, dx\, dy$$

$$= \int_{y=0}^{4} \frac{e^{2y}}{4-y} \int_{x=0}^{\sqrt{4-y}} x\, dx\, dy$$

$$= \int_{y=0}^{4} \frac{e^{2y}}{4-y} \cdot \left(\frac{x^2}{2} \Big|_0^{\sqrt{4-y}} \right) dy$$

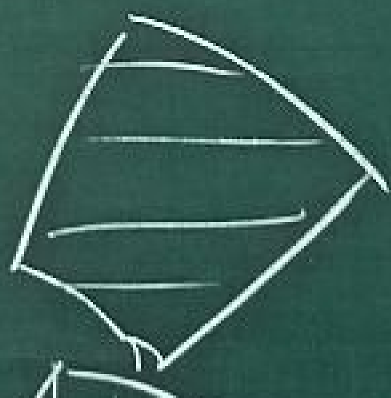
$$= \frac{1}{2} \int_0^4 e^{2y}\, dy$$

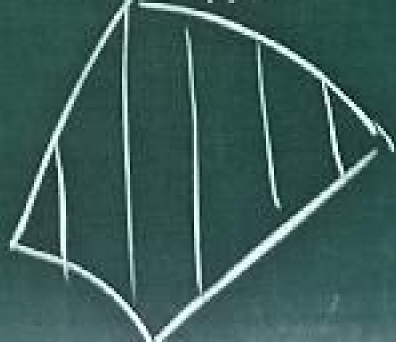
$$= \frac{1}{4} e^{2y} \Big|_0^4 = \frac{e^8 - 1}{4}$$

Integration in polar coordinate.

Eg 1 $\displaystyle\iint_R f(x,y)\,dA$

R = region enclosed by $\begin{cases} y = x \\ y = \sqrt{3}x \\ x^2+y^2=1 \\ x^2+y^2=4 \end{cases}$

Both  $(dx\,dy)$

and  $(dy\,dx)$ are complicated.

In this example

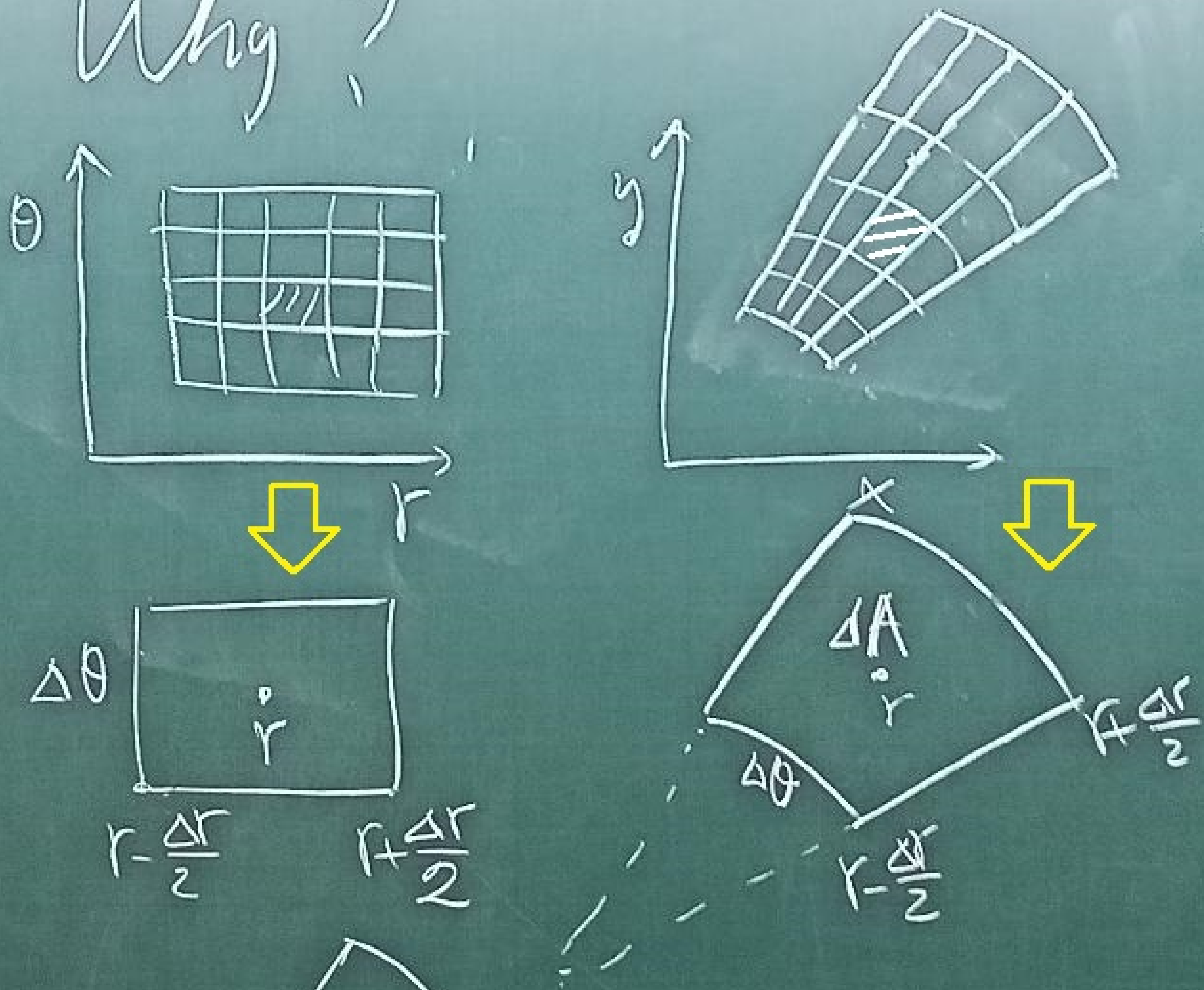
$$R = \left\{ 1 \le r \le 2,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{3} \right\}$$

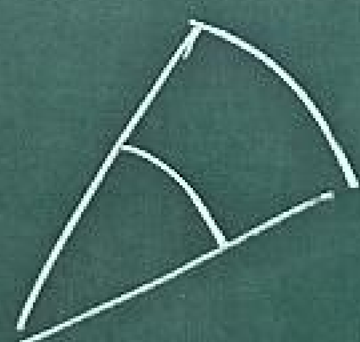
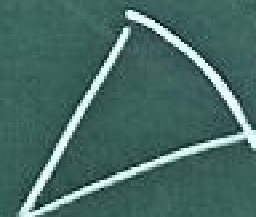
$$I = \int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{3}} f(\underset{\overset{\shortparallel}{x}}{r\cos\theta},\ \underset{\overset{\shortparallel}{y}}{r\sin\theta})\, dA$$

$$dA \overset{?}{=} dr\, d\theta$$

Ans: $dA = r\, dr\, d\theta$

Why?



ΔA =  − 

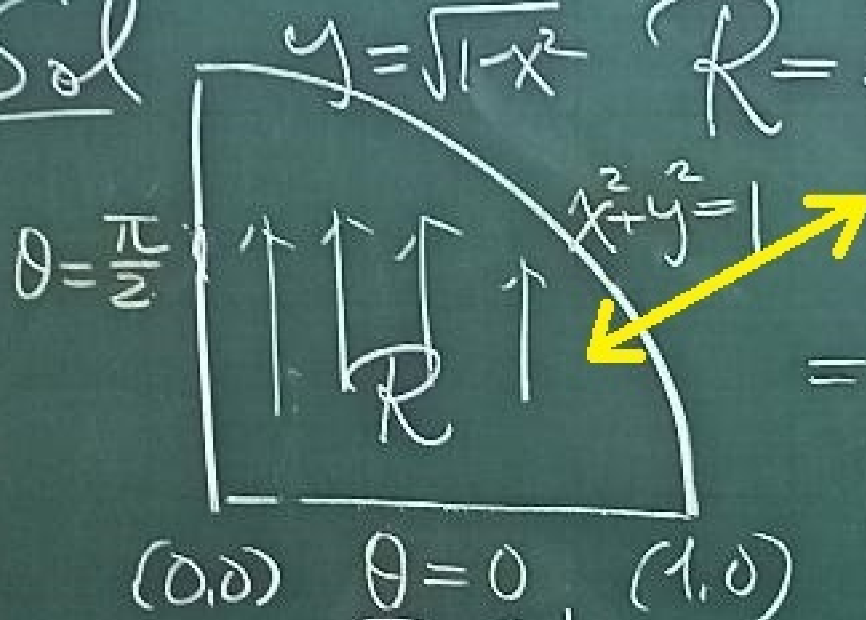
$$= \pi\left(r+\frac{\Delta r}{2}\right)^2\frac{\Delta\theta}{2\pi} - \pi\left(r-\frac{\Delta r}{2}\right)^2\frac{\Delta\theta}{2\pi}$$

$$= r\,\Delta r\,\Delta\theta$$

Integration in Polar Coordinates

Eg 2 $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}(x^2+y^2)\,dy\,dx$

Sol

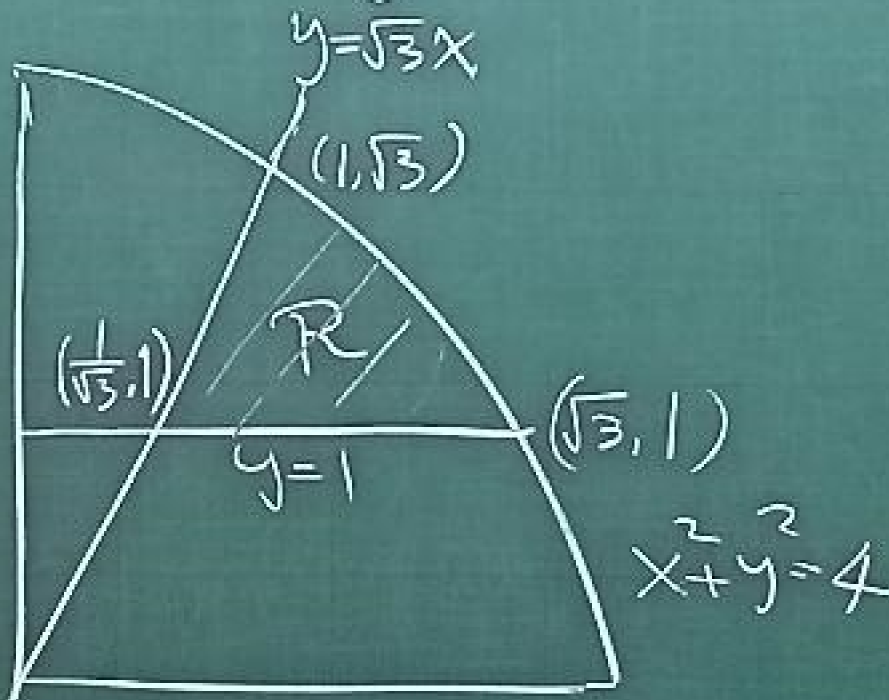


$$R=\left\{\begin{array}{l}0\le y\le\sqrt{1-x^2}\\ 0\le x\le 1\end{array}\right\}=\left\{\begin{array}{l}0\le r\le 1\\ 0\le\theta\le\frac{\pi}{2}\end{array}\right\}$$

$$I=\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{1} r^2\,\underbrace{(r\,dr\,d\theta)}_{=dA}$$

$$=\left(\int_0^1 r^3\,dr\right)\cdot\left(\int_0^{\frac{\pi}{2}} d\theta\right)=\frac{\pi}{8}$$

Eg 3 Find the area enclosed by $\begin{cases} y=\sqrt{3}x \\ y=1 \\ x^2+y^2=4 \end{cases}$ in 1st quadrant

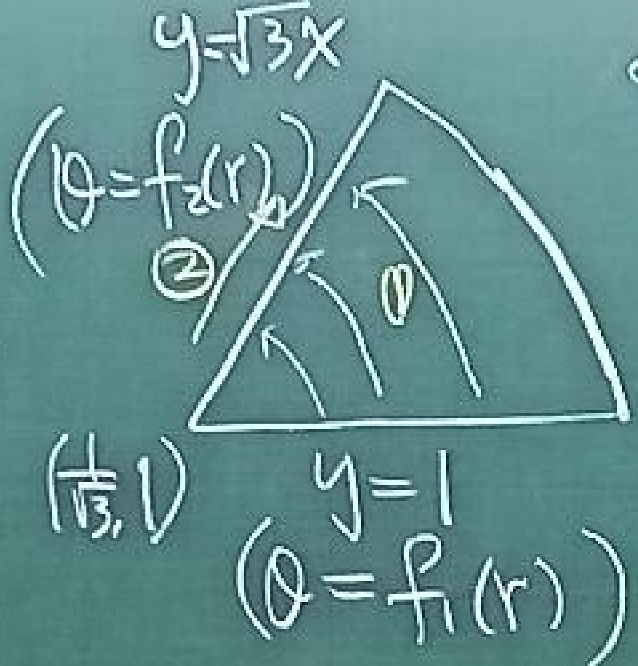


Sol: Method 1:

A = 

$$= \pi 2^2 \cdot \frac{\frac{\pi}{6}}{2\pi} - \frac{1}{2}\left(\sqrt{3}-\frac{1}{\sqrt{3}}\right)\cdot 1$$

Method 2: Polar Coordinate



$$R=\left\{ \begin{array}{l} f_1(r)\leq\theta\leq f_2(r) \\ \underset{\overset{\parallel}{\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2+1^2}}}{\frac{2}{\sqrt{3}}}\leq r\leq 2 \end{array} \right\}$$

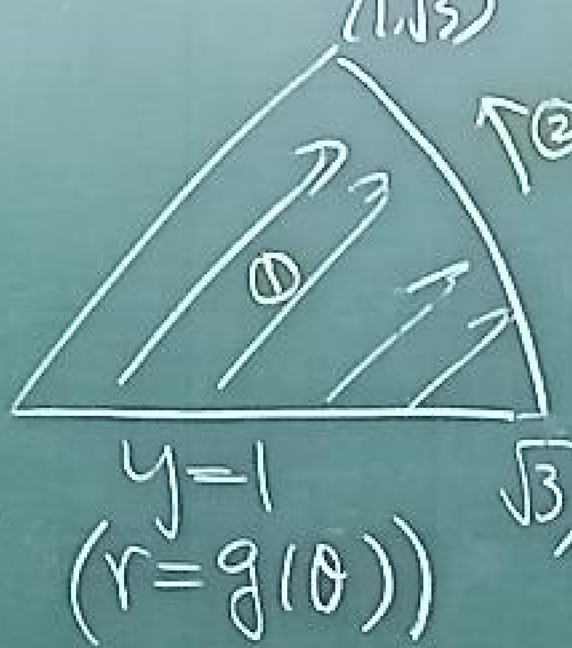
$$y=1 \Longleftrightarrow r\sin\theta=1 \Rightarrow \theta=\sin^{-1}\left(\frac{1}{r}\right)$$
$$=f_1(r)$$

$$y=\sqrt{3}x \Longleftrightarrow \theta=\frac{\pi}{3}=f_2(r)$$

$$I=\int_{r=\frac{2}{\sqrt{3}}}^{2}\int_{\theta=\sin^{-1}\left(\frac{1}{r}\right)}^{\frac{\pi}{3}} d\theta\, r\, dr \rightarrow \text{difficult}$$

$$=\int_{\frac{2}{\sqrt{3}}}^{2} r\left(\frac{\pi}{3}-\sin^{-1}\left(\frac{1}{r}\right)\right)dr=?$$

Method 3



$$R = \left\{ \begin{array}{l} g(\theta) \le r \le 2 \\[2ex] \underset{\substack{\| \\ \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)}}{\frac{\pi}{6}} \le \theta \le \underset{\substack{\| \\ \tan^{-1}(\sqrt{3})}}{\frac{\pi}{3}} \end{array} \right\}$$

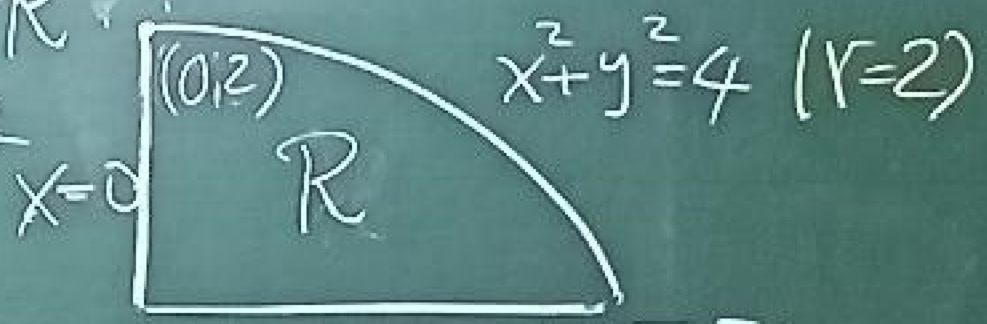
$$y = 1 \Leftrightarrow r\sin\theta = 1 \Rightarrow r = \frac{1}{\sin\theta} = g(\theta)$$

$$I = \int_{\theta=\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{r=\frac{1}{\sin\theta}}^{2} r \, dr \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(2 - \frac{1}{2\sin^2\theta}\right) d\theta$$

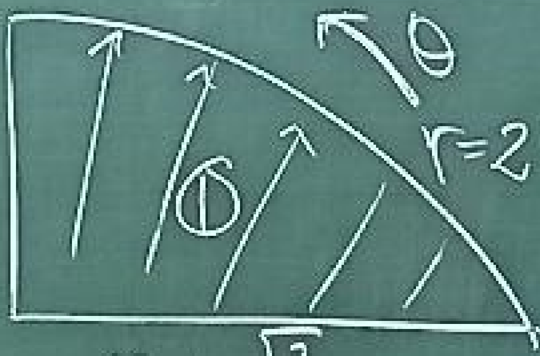
$$= \left(2\theta + \frac{\cot\theta}{2}\right)\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{\pi}{3} + \frac{1}{2}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)$$

Eg4: Evaluate $\iint_R f(x,y)\,dA$ in polar coordinate where



$(r=\sqrt{2}) \leftrightarrow (0,\sqrt{2})$ $y=\sqrt{2}$ $(\sqrt{2},\sqrt{2})$ $(\theta=\frac{\pi}{4})$

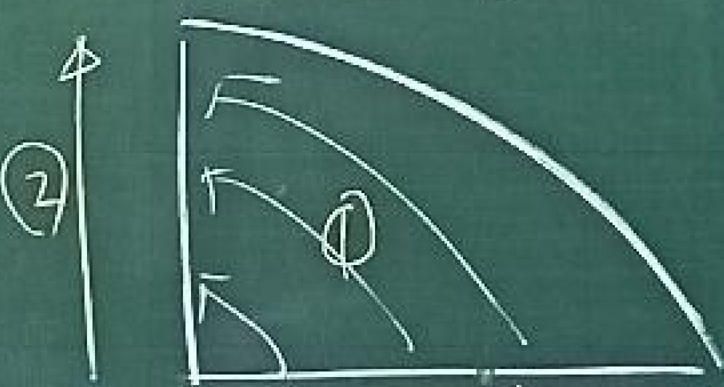
Sol: Method ⓐ $r\,dr\,d\theta$



$r=\frac{\sqrt{2}}{\sin\theta}$

$$I=\int_{\theta=\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{r=\frac{2}{\sin\theta}}^{2} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Method ⓑ $d\theta\,r\,dr$

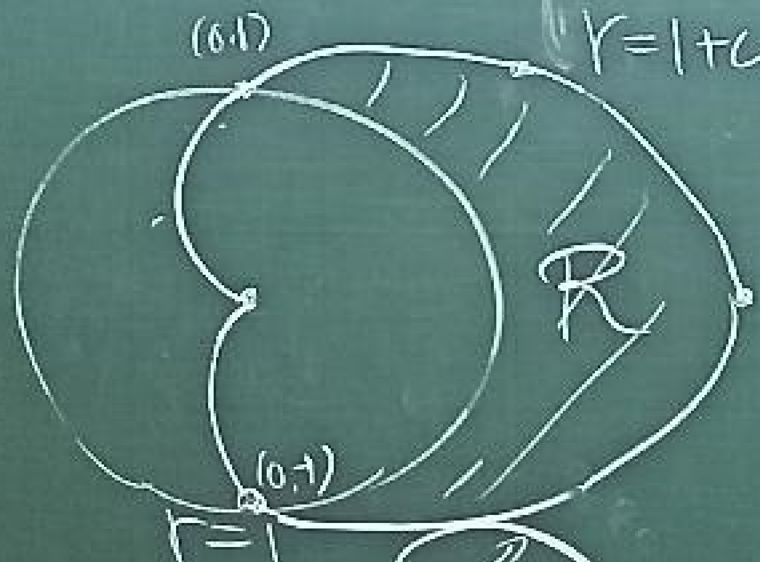


$\theta=\sin^{-1}\left(\frac{\sqrt{2}}{r}\right)$

$$I=\int_{r=\sqrt{2}}^{2}\int_{\theta=\sin^{-1}\left(\frac{\sqrt{2}}{r}\right)}^{\frac{\pi}{2}} f(r\cos\theta, r\sin\theta)\,d\theta\,r\,dr$$

Eg 5 Evaluate $\iint_R g(r,\theta)\,dA$

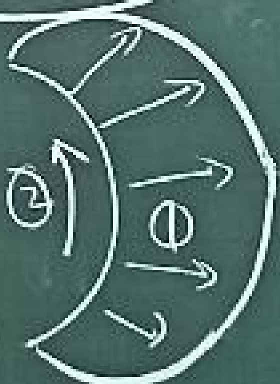
where R is the region $\begin{cases} \text{inside } r = 1+\cos\theta \\ \text{outside } r = 1 \end{cases}$



$r = 1+\cos\theta \Rightarrow \cos\theta = r-1$

$\Rightarrow \theta = \begin{cases} \cos^{-1}(r-1) & \text{if } \theta \in [0, \frac{\pi}{2}] \\ -\cos^{-1}(r-1) & \text{if } \theta \in [\frac{-\pi}{2}, 0] \end{cases}$

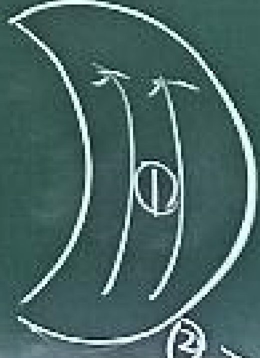
(a) $r\,dr\,d\theta$: (usually preferred)



$$I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_{r=1}^{1+\cos\theta} g(r,\theta)\, r\,dr\,d\theta$$

(b) $d\theta\, r\,dr$

(usually not good)



$$I = \int_{r=1}^{2} \int_{-\cos^{-1}(r-1)}^{\cos^{-1}(r-1)} g(r,\theta)\, d\theta\, r\,dr$$

Remark

$r\,dr\,d\theta$ is usually a better choice than $d\theta\,r\,dr$ since most curves in polar coordinates appear naturally in the form $r = f(\theta)$, which is needed in $\int_{(*)}^{(*)} r\,dr$