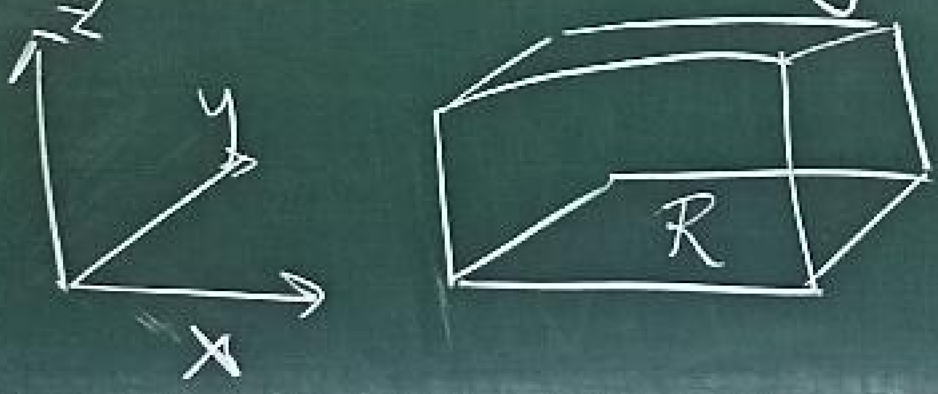
Double Integral

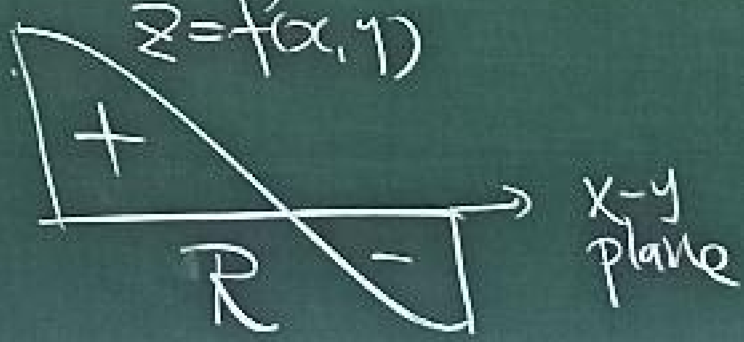
Def: Let $R = [a,b] \times [c,d]$

$= \{ a \le x \le b,\ c \le y \le d \}$

$$\iint_R f(x,y)\, dA$$

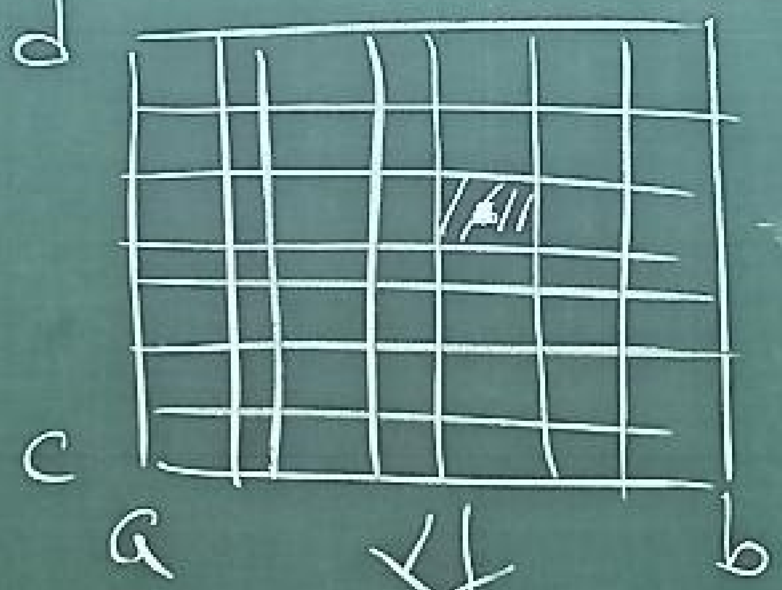
= Signed Volume between $z = f(x,y)$ graph and x-y plane over the region R.

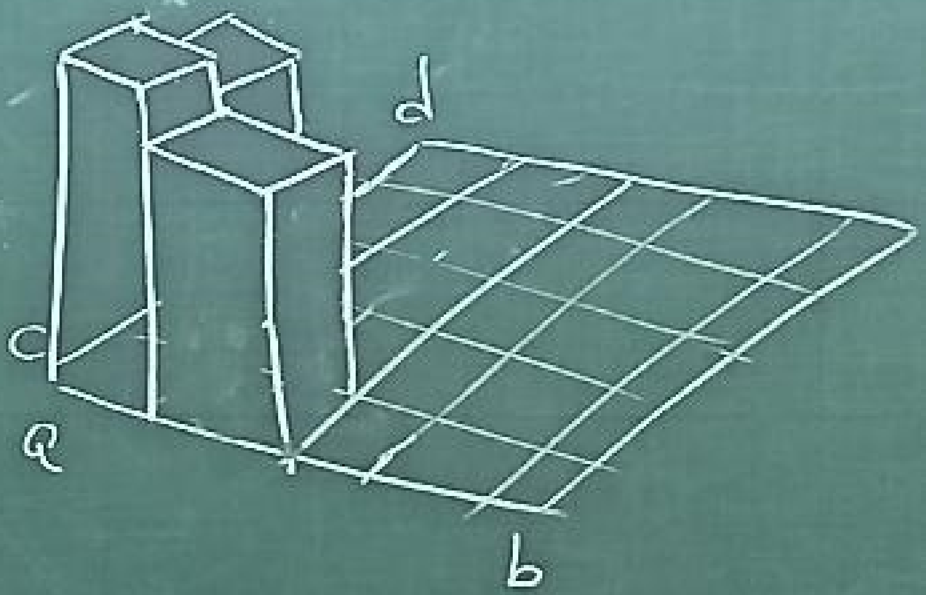




In other words,

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k,y_k)\,\Delta A_k$$





$\Downarrow$

R_k : (x_k, y_k) — kth rectangle

Area $\stackrel{\text{def}}{=} \Delta A_k$

$$P = \left\{ \begin{array}{l} a = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_M = b \\ c = \bar{y}_0 < \bar{y}_1 < \cdots < \bar{y}_N = d \end{array} \right\}$$

$$\|P\| = \max_{\text{all } i,\, j} \{ \underbrace{\Delta x_i}_{\bar{x}_i - \bar{x}_{i-1}}, \underbrace{\Delta y_j}_{\bar{y}_j - \bar{y}_{j-1}} \}$$

$(x_k, y_k) \in R_k$

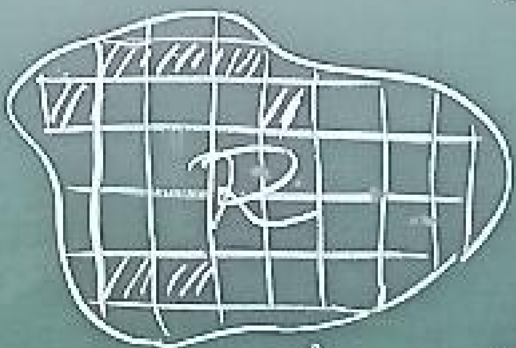
In general, if $f(x,y)$ is continuous on R, then $\iint_R f(x,y)\,dA$ exists.
(f is integrable on R)

Fubini's Theorem.

If f is continuous on $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

General Region R

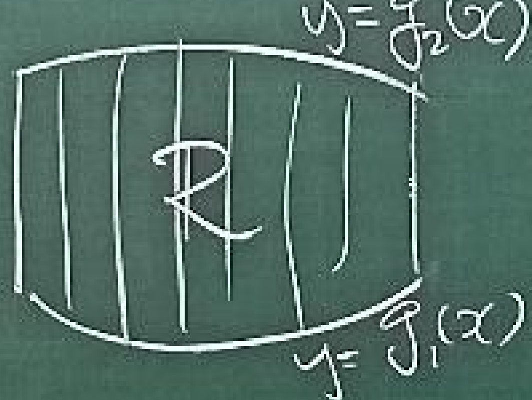


$$\iint_R f(x,y)\,dA$$

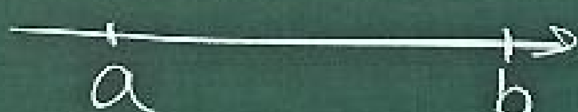
$$= \lim_{\|P\|\to 0} \sum_{\substack{k=1 \\ R_k \subset R}}^{n} f(x_k, y_k)\,\Delta A_k$$

Special cases

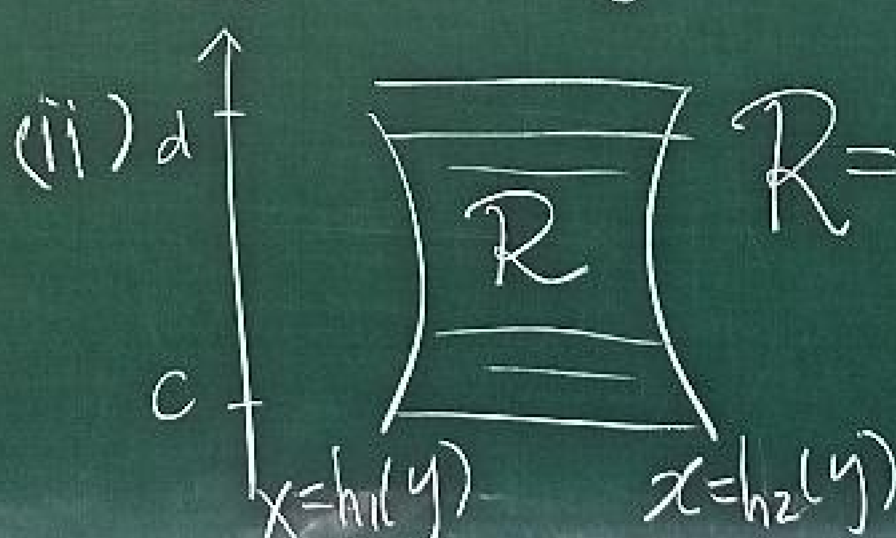
(i)



$$R = \left\{ \begin{array}{l} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$$



(ii)



$$R = \left\{ \begin{array}{l} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{array} \right\}$$

Fubini's Theorem (Strong form)

If f is cont. on R and

(i) $R = \{a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Then $\iint_R f(x,y)\,dA = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$

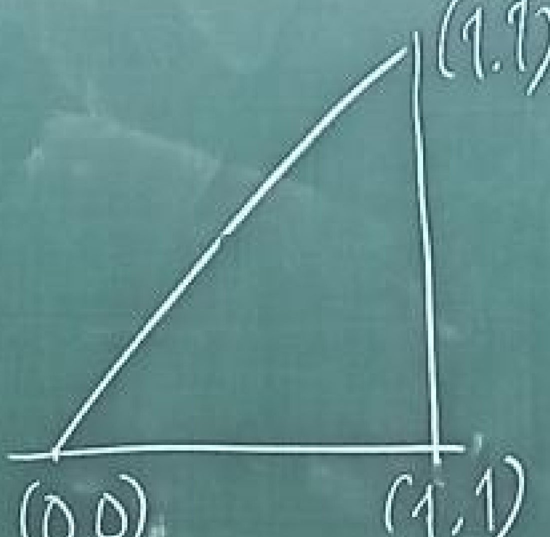
or

(ii) $R = \{c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$

Then $\iint_R f(x,y)\,dA = \int_{y=c}^{d} \int_{x=h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$

Eg 1 R = region enclosed by

$x=y$, $x=1$ and $y=0$



$$\iint_R \frac{\sin x}{x}\,dA = ?$$

Sol: $R = \left\{0\le x\le 1,\ 0\le y\le x\right\}^{(i)}$ 

$\quad = \left\{0\le y\le 1,\ y\le x\le 1\right\}_{(ii)}$ 

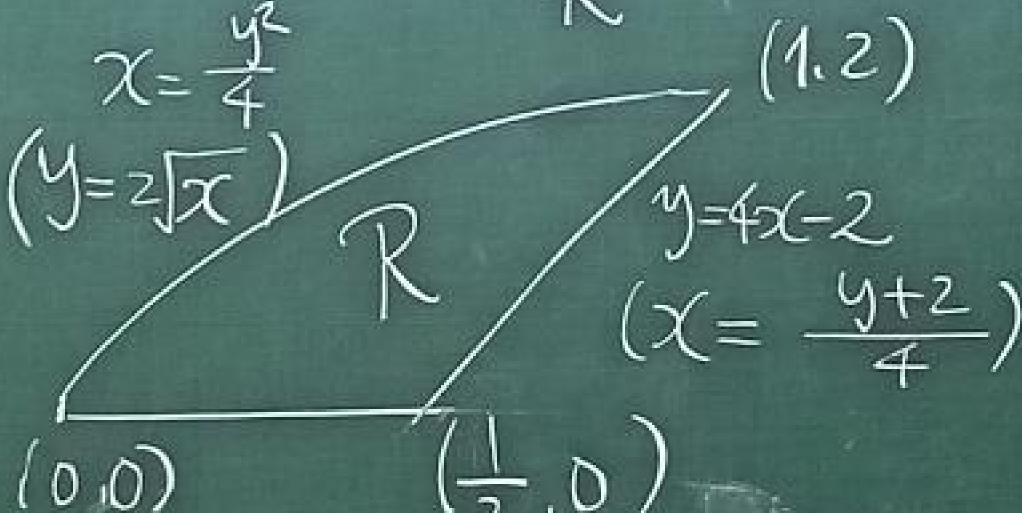
$$(ii) = \int_{y=0}^{1}\int_{x=y}^{1} \frac{\sin x}{x}\,dx\,dy = ?$$

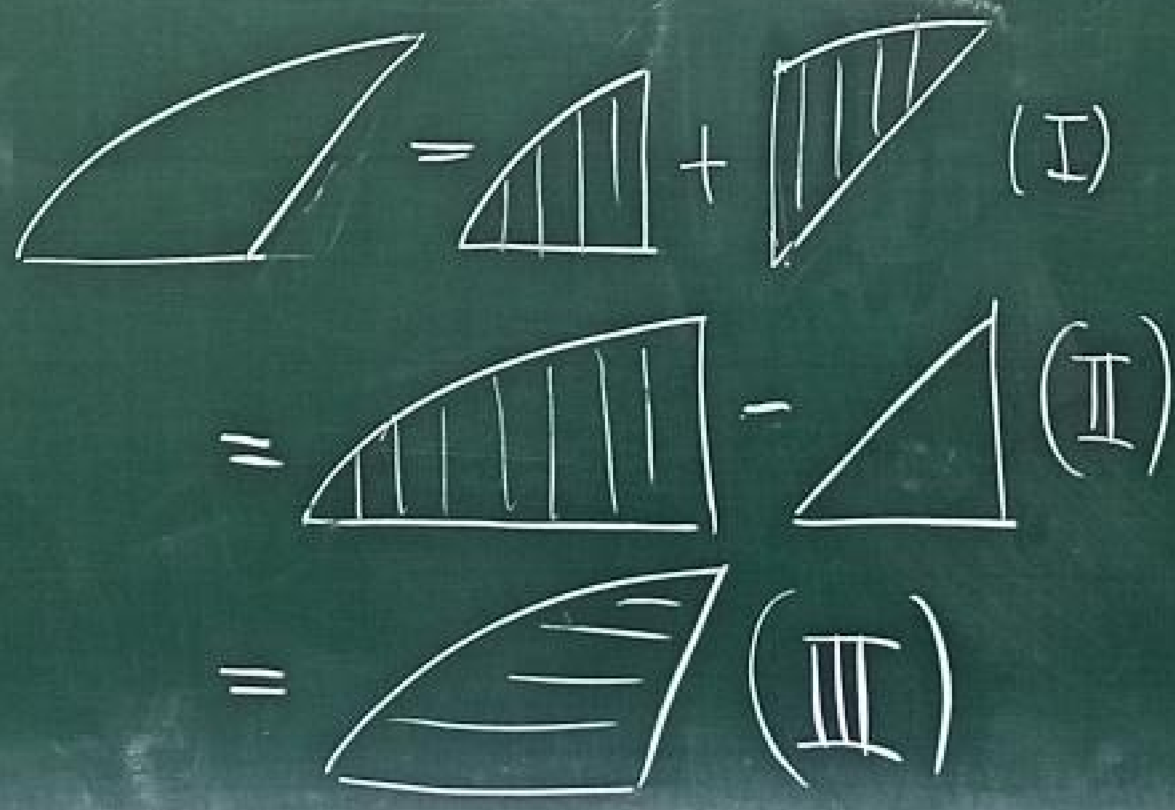
$$(i) = \int_{x=0}^{1}\int_{y=0}^{x} \frac{\sin x}{x}\,dy\,dx$$

$$= \int_0^1 \left(\frac{\sin x}{x}\int_{y=0}^{x} 1\,dy\right)dx = -\cos x\Big|_0^1 = 1-\cos 1$$

Eg2: Find the area of the region enclosed by $\begin{cases} y=0 \\ y=4x-2 \\ x=\frac{y^2}{4} \\ (y>0) \end{cases}$

Sol: $A=\iint_R 1\,dA$





(I): exercise.

$$(\text{II}) = \int_{x=0}^{1}\int_{y=0}^{2\sqrt{x}} 1\,dy\,dx - \frac{1}{2}\cdot\frac{1}{2}\cdot 2$$

$$= \int_0^1 2\sqrt{x}\,dx - \frac{1}{2} = \frac{5}{6}$$

$$(\text{III}) = \int_{y=0}^{2}\int_{x=\frac{y^2}{4}}^{\frac{y+2}{4}} 1\,dx\,dy$$

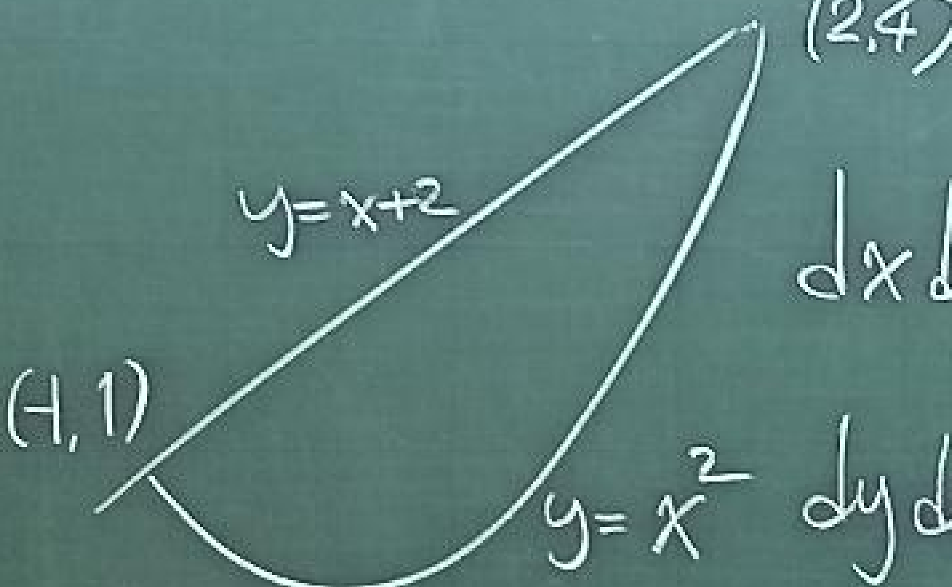
$$= \int_{y=0}^{2}\left(\frac{y+2}{4} - \frac{y^2}{4}\right)dy$$

$$= \int_0^2 -\frac{y^2}{4} + \frac{y}{4} + \frac{1}{2}\,dy$$

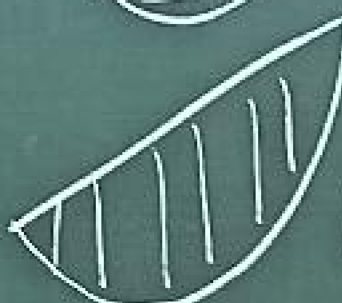
$$= -\frac{y^3}{12} + \frac{y^2}{8} + \frac{y}{2}\Big|_{y=0}^{2}$$

$$= \frac{5}{6}$$

Eg 3. Find area of the region enclosed by $\begin{cases} y = x^2 \\ y = x+2 \end{cases}$



dx dy: 

dy dx:  (Simpler)

$$A = \int_{x=-1}^{2} \int_{y=x^2}^{x+2} dy\, dx = \int_{-1}^{2} x+2-x^2\, dx$$

$$= \frac{x^2}{2} + 2x - \frac{x^3}{3} \Big|_{-1}^{2} = \frac{9}{2}$$