

Partial Derivative with Constraint

Ex1. Find $\frac{\partial w}{\partial x}$ if $\begin{cases} w = x^2 + y^2 + z^2 \\ z = x^2 + y^2 \end{cases}$

Sol total 4 variables + 2 eqns.

3 variables + 1 eqn

$$\begin{aligned} x + y - z &= 5 \\ \Rightarrow x &= f(y, z) \quad \begin{matrix} \nearrow \\ \text{independent} \\ \text{variables} \end{matrix} \\ & \quad (\text{or } y = g(z, x), z = h(x, y)) \end{aligned}$$

3 Variables + 2 eqns

$$\begin{aligned} &\begin{cases} x + y - z = 5 \\ x - y + 2z = 6 \end{cases} \\ \Rightarrow &\begin{cases} x = f_1(z) \\ y = f_2(z) \end{cases} \text{ or } \begin{cases} y = g_1(x) \\ z = g_2(x) \end{cases} \text{ or } \begin{cases} z = h_1(y) \\ x = h_2(y) \end{cases} \end{aligned}$$

4 variables 2 eqns

$\frac{\partial \omega}{\partial x} \rightarrow$ dependent ω independent x
(ω, y) (x, z)
or (ω, z) (x, y)

2 possibilities

$$(A): \begin{cases} \omega = f_1(x, z) \\ y = g(x, z) \end{cases} \quad (B): \begin{cases} \omega = f_2(x, y) \\ z = h(x, y) \end{cases}$$

$$\frac{\partial \omega}{\partial x} = \partial_x f_1(x, z) \quad \text{or} \quad \frac{\partial \omega}{\partial x} = \partial_x f_2(x, y)$$

They are different! Need to specify $\left(\frac{\partial \omega}{\partial x}\right)_y$ or $\left(\frac{\partial \omega}{\partial x}\right)_z$

$\left(\frac{\partial \omega}{\partial x}\right)$ alone, is not a valid question

$$\left(\frac{\partial \omega}{\partial x}\right)_y: \begin{cases} \omega = f_2(x, y) = x^2 + y^2 + (x^2 + y^2)^2 \\ z = h(x, y) = x^2 + y^2 \end{cases}$$

$$\left(\frac{\partial \omega}{\partial x}\right)_z: \begin{cases} \omega = f_1(x, z) = z + z^2 \\ y = g(x, z) = \pm \sqrt{z - x^2} \end{cases}$$

Ex 2. Find $\left(\frac{\partial \omega}{\partial x}\right)_y$ at $(x, y, z) = (2, -1, 1)$
if $\begin{cases} \omega = x^2 + y^2 + z^2 \\ z^3 - xy + yz + y^3 = 1 \end{cases}$

Ans $\left(\frac{\partial \omega}{\partial x}\right)_y \Rightarrow \begin{cases} \omega = \omega(x, y) \\ z = z(x, y) \end{cases}$

$$\Rightarrow \omega_x = 2x + 2z z_x \quad (1)$$

$$3z^2 z_x - y + yz_x = 0 \quad (2)$$

$$(2) \Rightarrow \left(\frac{\partial z}{\partial x}\right)_y (2, -1, 1) = \frac{y}{3z^2 + y} = -\frac{1}{2}$$

$$(1) \Rightarrow \left(\frac{\partial \omega}{\partial x}\right)_y (2, -1, 1) = 2 \cdot 2 + 2 \cdot 1 \cdot \left(-\frac{1}{2}\right) = \underline{\underline{3}}$$

Ex 3 Find $(\frac{\partial w}{\partial x})_{y,z}$ if $\begin{cases} w = x^2 + y - z + \sin t \\ x + y = t \end{cases}$

Sol: 5 variables + 2 eqns

$\Rightarrow$ 2 dependent + 3 independent

$$(\frac{\partial w}{\partial x})_{y,z} = \begin{matrix} w = w(x, y, z) \\ t = t(x, y, z) \end{matrix}$$

$$\Rightarrow w_x = 2x + 0 - 0 + \cos(x+y) \neq$$

General Procedure: Find $(\frac{\partial z}{\partial x})_y$ if $\begin{cases} f(x, y, z, w) = 0 \\ g(x, y, z, w) = 0 \end{cases}$

Ans: $z = z(x, y), w = w(x, y)$

$$(\frac{\partial}{\partial x})_y \Rightarrow \begin{cases} f_x + f_z z_x + f_w w_x = 0 \\ g_x + g_z z_x + g_w w_x = 0 \end{cases}$$

$\Rightarrow$ 2 unknowns (z_x, w_x) + 2 linear eqns

$\Rightarrow$ Solve for z_x and w_x

Eg4. Find $\left(\frac{\partial(PV)}{\partial T}\right)_{n,V}$

(*) if $PV = nRT$, $R = \text{absolute constant}$

Sol. closed system: $n = \text{fixed}$

4 variables + 1 eqn (*)
(P, V, n, T)

$$\left(\frac{\partial(PV)}{\partial T}\right)_{n,V} \Rightarrow P = P(n, V, T)$$

$$\frac{\partial(P(n, V, T) \cdot V)}{\partial T} = P_T \cdot V$$

$$(*) \Rightarrow P_T V = nR \quad \#$$

Eg 5: Find $\left(\frac{\partial(PV)}{\partial T}\right)_n$ if $PV = nRT$

Sol. $\left(\frac{\partial(PV)}{\partial T}\right)_n \Rightarrow \begin{matrix} P = f(n, T) \\ V = g(n, T) \end{matrix}$

Explicit form of f, g are not known

except $f(T, n)g(T, n) = nRT$

$$\Rightarrow \left(\frac{\partial(PV)}{\partial T}\right)_n = \frac{\partial(f(T, n)g(T, n))}{\partial T}$$

$$= \frac{\partial}{\partial T}(nRT) = nR \quad (\text{Hw})$$