Recall:

Let $F(t) = f(a+th, b+tk)$

$$\Rightarrow F^{(\ell)}(t) = (h\partial_x + k\partial_y)^\ell f(a+th, b+tk)$$

or symbolically $\left(\frac{d}{dt}\right)^\ell = (h\partial_x + k\partial_y)^\ell$

Similarly for $g(x,y,z)$

Let $G(t) = g(x_0+t\Delta x, y_0+t\Delta y, z_0+t\Delta z)$

$\Delta x = x - x_0$, $\Delta y = y - y_0$, $\Delta z = z - z_0$

$$\left(\frac{d}{dt}\right)^\ell G = (\Delta x\,\partial_x + \Delta y\,\partial_y + \Delta z\,\partial_z)^\ell g$$

$$\tilde{P}_n = \sum_{\ell=0}^{n} \frac{(\Delta x\partial_x + \Delta y\partial_y + \Delta z\partial_z)^\ell}{\ell!} g(x_0, y_0, z_0)$$

$$\tilde{R}_n = \frac{(\cdots)^{n+1}}{(n+1)!} g(x_0+c\Delta x, y_0+c\Delta y, z_0+c\Delta z) \quad 0<c<1$$

Eg1 Error of linearization for $f(x,y)$

$$L(x,y) = f(x_0,y_0) + \underbrace{\partial_x f(x_0,y_0)(x-x_0) + \partial_y f(x_0,y_0)(y-y_0)}_{(\Delta x\partial_x + \Delta y \partial_y)^1 f(x_0,y_0)}$$

$$(= \tilde{P}_1(x,y))$$

$$f(x,y) = \tilde{P}_1(x,y) + \tilde{R}_1(x,y)$$

$$\therefore \text{error} = f(x,y) - L(x,y) = \tilde{R}_1(x,y)$$

$$= \frac{(\Delta x\partial_x + \Delta y\partial_y)^2}{2!} f(x_0 + c\Delta x,\ y_0 + c\Delta y)$$

$$\therefore |\text{error}| \leq \frac{1}{2}\Big(\underbrace{|\Delta x^2 \partial_x^2 f| + |2\Delta x\Delta y\,\partial_x\partial_y f| + |\Delta y^2 \partial_y^2 f|}_{\text{at } (x_0 + c\Delta x,\ y_0 + c\Delta y)}\Big)$$

$$\leq \frac{M}{2}(|\Delta x| + |\Delta y|)^2, \text{ if } |f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$$

Similarly

$$|f(x,y,z)-L(x,y,z)| \leq \frac{M}{2}(|\Delta x|+|\Delta y|+|\Delta z|)^2$$

if all |2nd derivatives| $\leq M$ in the region

Eg2. Local min/max of f

$$f(x,y)=\underbrace{f(x_0,y_0)}_{l=0}+\underbrace{\Delta_1}_{l=1}+\underbrace{\Delta_2}_{l=2}+\underbrace{\Delta_3}_{\tilde{R}_2}$$

At a critical point, $\Delta_1=0$

$\Delta_2=$ leading term: $O(\Delta x^2, \Delta x\Delta y, \Delta y^2)$

$$\Delta_3=\tilde{R}_2(x,y)=\frac{(\Delta x\partial_x+\Delta y\partial_y)^3}{3!}f(x_0+c\Delta x, y_0+c\Delta y)$$

$$\therefore |\Delta_3| \leq \frac{1}{3!}\left(M|\Delta x|^3+3M|\Delta x|^2|\Delta y|+3M|\Delta x||\Delta y|^2+M|\Delta y|^3\right)$$

$$=\frac{M}{3!}(|\Delta x|+|\Delta y|)^3$$

if $|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}| \leq M$