Recall: Find local extremes of $f(x,y)$ subject to $g(x,y)=0$

$\Rightarrow$ Solve for (x_0,y_0), λ from

$$\begin{cases} g(x_0,y_0)=0 \\ \nabla f(x_0,y_0)=\lambda \nabla g(x_0,y_0) \end{cases}$$

$\underline{\underline{3}}$ equations, $\underline{\underline{3}}$ unknowns

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§ Find local extremes of $f(x,y,z)$ subject to the constraint $g(x,y,z)=0$

$\Rightarrow$ Solve for (x_0,y_0,z_0), λ from

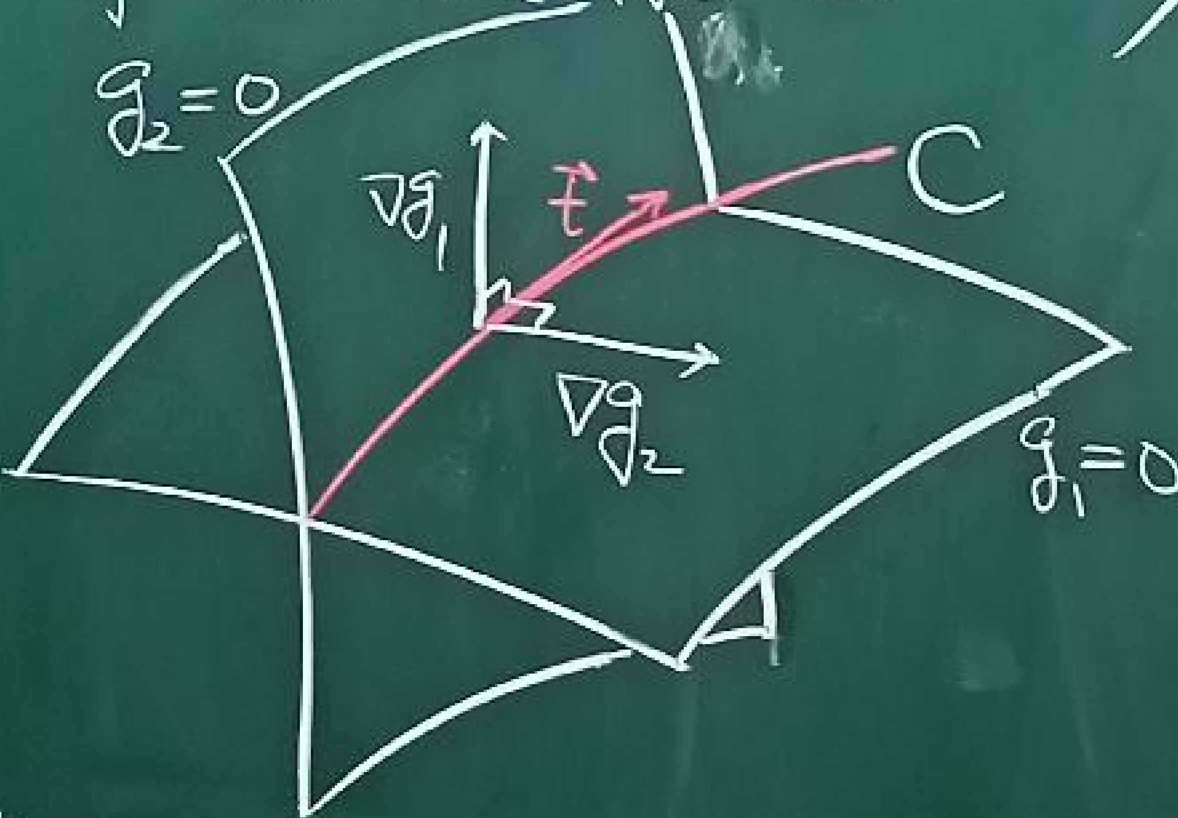
$$\begin{cases} g(x_0,y_0,z_0)=0 \\ \nabla f(x_0,y_0,z_0)=\lambda \nabla g(x_0,y_0,z_0) \end{cases}$$

$\underline{\underline{4}}$ equations, $\underline{\underline{4}}$ unknowns

§ Find local extremes of $f(x,y,z)$ subject to

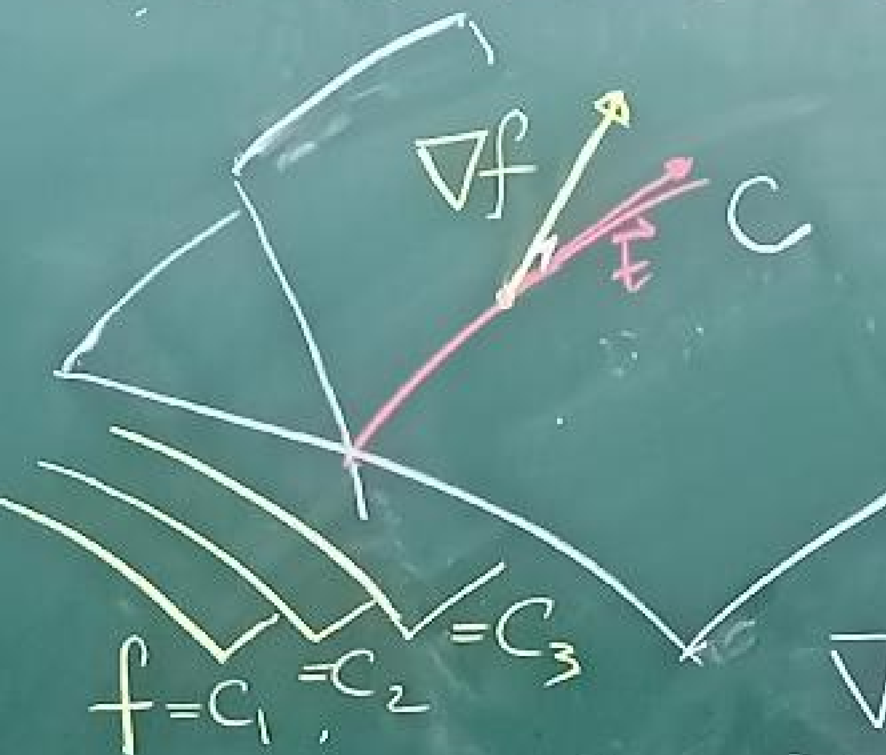
$$\begin{cases} g_1(x,y,z)=0 \\ g_2(x,y,z)=0 \end{cases}$$

(Find local extremes of f on a curve C)



(Figure 14.38)

At a local extreme point



$\vec{t}$: tangent vector of C at (x_0, y_0, z_0)

$C \subseteq \{g_1 = 0\}$

$\nabla g_1(x_0, y_0, z_0) \perp \{g_1 = 0\}$

$\Rightarrow \nabla g_1(x_0, y_0, z_0) \perp \vec{t}$

Similarly $\nabla g_2(x_0, y_0, z_0) \perp \vec{t}$

Moreover, at a local extreme point

C is tangent to $\{f(x,y,z) = c\}$

$\Rightarrow \vec{t} \perp \nabla f, \nabla g_1, \nabla g_2$ at (x_0, y_0, z_0)

$\Rightarrow \nabla f, \nabla g_1, \nabla g_2$ are Co-plane at (x_0, y_0, z_0)

$\Rightarrow \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ at (x_0, y_0, z_0)

Summary:

Solve (x_0, y_0, z_0), λ_1, λ_2 from

$$g_1(x_0, y_0, z_0) = 0$$

$$g_2(x_0, y_0, z_0) = 0$$

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 \text{ at } (x_0, y_0, z_0)$$

$$\left(\begin{aligned} \partial_x f &= \lambda_1 \partial_x g_1 + \lambda_2 \partial_x g_2 \\ \partial_y f &= \lambda_1 \partial_y g_1 + \lambda_2 \partial_y g_2 \\ \partial_z f &= \lambda_1 \partial_z g_1 + \lambda_2 \partial_z g_2 \end{aligned} \right)$$

$\underline{\underline{5}}$ equations, $\underline{\underline{5}}$ unknowns

Eg1 Find nearest point to origin on the curve $\begin{cases} x+y+z=1 \\ x^2+y^2=1 \end{cases}$

Solve: Minimize $f(x,y,z)=x^2+y^2+z^2$

subject to $\begin{cases} g_1 = x+y+z=1 \\ g_2 = x^2+y^2=1 \end{cases}$

$$x+y+z=1 \quad \text{——} \ ①$$

$$x^2+y^2 = 1 \quad \text{——} \ ②$$

$$2x = \lambda_1 1 + \lambda_2 \cdot 2x \quad \text{—} \ ③$$

$$2y = \lambda_1 1 + \lambda_2 \cdot 2y \quad \text{—} \ ④$$

$$2z = \lambda_1 \cdot 1 \quad \text{—} \ ⑤$$

$⑤ \to ③, ④$

$$\Rightarrow \begin{aligned} x(1-\lambda_2) &= z \\ y(1-\lambda_2) &= z \end{aligned}$$

$\Rightarrow$ ⓐ $\lambda_2 = 1,\ z = 0$

or ⓑ $\lambda_2 \neq 1,\ x = y$

<u>Case ⓐ</u>

$①② \Rightarrow \begin{cases} x+y=1 \\ x^2+y^2=1 \end{cases}$

$\Rightarrow (x,y,z) = \boxed{\begin{matrix}(1,0,0)\\(0,1,0)\end{matrix}}$

Case ⓑ $(x=y)$

$② \Rightarrow (x,y) = \pm\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$z = 1-x-y = 1 \mp \sqrt{2}$

$(1,0,0), (0,1,0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1-\sqrt{2}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 1+\sqrt{2}\right)$

$f = \underbrace{1,\ 1}_{\text{abs min}} \qquad \underbrace{4-2\sqrt{2}}_{\text{local max}}, \qquad \underbrace{4+2\sqrt{2}}_{\text{abs max}}$

Taylor polynomial of $f(x,y)$ centered at (a,b)

$x = a+h$
$y = b+k$

Let $F(t) = f(a+th,\, b+tk)$

$h = x-a,\ k = y-b,\ t \in \mathbb{R}$

$F(0) = f(a,b),\ F(1) = f(x,y)$

Assume f and all partial derivatives of f are continuous everywhere.

Taylor's Thm for $F(t)$

$$F(t) = P_n(t) + R_n(t)$$

(Want $F(1) = P_n(1) + R_n(1)$.) (*)

where $P_n(t) = \sum_{\ell=0}^{n} \frac{F^{(\ell)}(0)}{\ell!} t^\ell$

$$R_n(t) = \frac{F^{(n+1)}(c)}{(n+1)!} t^{n+1}$$

for some c between 0 and t

Take $t = 1$

$$P_n(1) = \sum_{\ell=0}^{n} \frac{F^{(\ell)}(0)}{\ell!}$$

$$R_n(1) = \frac{F^{(n+1)}(c)}{(n+1)!}, \quad 0 < c < 1$$

$$F^{(k)}(0) = ?$$

$$F'(t) = \frac{d}{dt} f(x(t), y(t))$$

where $x(t) = a + th$, $y(t) = b + tk$

$$= f_x(x(t), y(t)) \cdot \underbrace{x'(t)}_{= h} + f_y(x(t), y(t)) \cdot \underbrace{y'(t)}_{= k}$$

$$= (h\partial_x + k\partial_y) f(x(t), y(t))$$

$$\therefore \text{"} \frac{d}{dt} F = (h\partial_x + k\partial_y) f \text{"}$$

$$F''(t) = \frac{d}{dt} F'(t) = \frac{d}{dt} \overbrace{(h\partial_x + k\partial_y) f}^{= f_1(x(t), y(t))}$$

$$= \partial_x(\underbrace{(h\partial_x + k\partial_y) f}_{f_1}) \cdot \frac{dx}{dt} + \partial_y((h\partial_x + k\partial_y) f) \cdot \frac{dy}{dt}$$

$$= h\partial_x((h\partial_x + k\partial_y) f) + k\partial_y((h\partial_x + k\partial_y) f)$$

$$= (h\partial_x + k\partial_y)^2 f(x(t), y(t))$$

In Summary

$$F^{(\ell)}(t) = (h\partial_x + k\partial_y)^{\ell} f(a+th,\, b+tk)$$

for example:

$$F^{(3)} = (h^3\partial_x^3 + 3h^2k\,\partial_x^2\partial_y + 3hk^2\,\partial_x\partial_y^2 + k^3\partial_y^3) f$$

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$$\overset{(*)}{\Longrightarrow} f(x,y) = \tilde{P}_n(x,y) + \tilde{R}_n(x,y)$$

$$\tilde{P}_n(x,y) = P_n(1) = \sum_{\ell=0}^{n} \frac{(h\partial_x + k\partial_y)^{\ell}}{\ell!} f \Big|_{(a,b)}$$

$$\tilde{R}_n(x,y) = R_n(1)$$

$$= \frac{(h\partial_x + k\partial_y)^{n+1}}{(n+1)!} f(a+ch,\, b+ck)$$

$$0 < c < 1$$