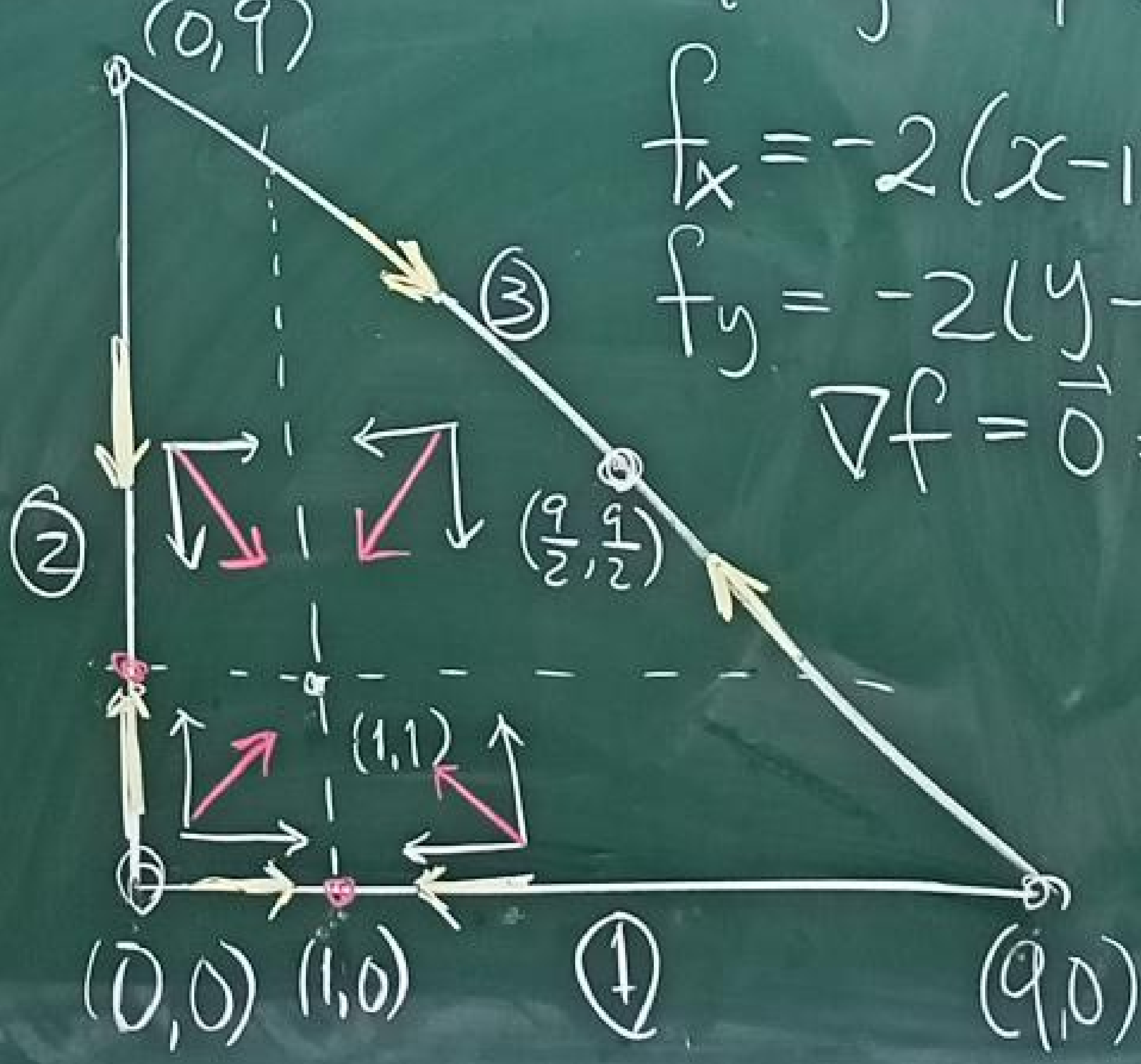
Eg2: Find absolute extremes of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the region bounded by
$$\begin{cases} x = 0 \\ y = 0 \\ y = 9 - x \end{cases}$$




$f_x = -2(x-1)$

$f_y = -2(y-1)$

$\nabla f = \vec{0} \Rightarrow (1,1)$

Method 1 (textbook)

Critical Point $= (1.1)$.

Compare $f(1.1)$ with all values of $f$ on boundary

Method 2 (Gradient Analysis)

Near $(1,1)$

local max $(1,1)$

on ① $= \begin{cases} y=0 \\ 0 \le x \le 9 \end{cases}$

$(1,0)$

local max $= (1,0)$

local min $= (0.0), (9,0)$

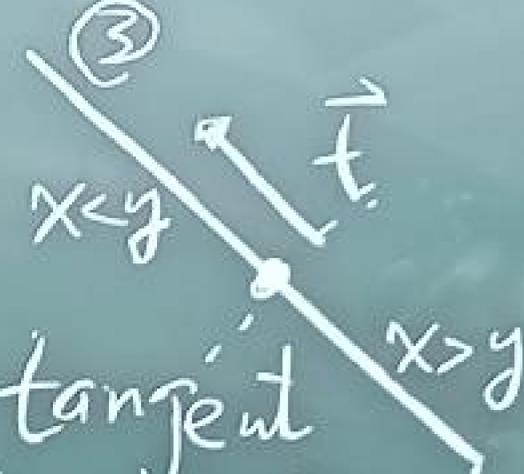
on ②:

$(0,1)$

local max $= (0,1)$

local min $= (0,0), (0,9)$

On ③ $= \begin{cases} x+y=9 \\ 0 \le x \le 9 \end{cases}$




Let $\vec{t} = (-1, 1)$ = a tangent vector of ③

$\nabla f \cdot \vec{t} \begin{matrix} >0 \\ <0 \end{matrix}$ ?

$= (-2(x-1), -2(y-1)) \cdot (-1, 1)$

$= \sqrt{2}(x-y) \begin{cases} >0 & \text{if } x>y \\ <0 & \text{if } x<y \end{cases}$


On ③, local max $= (\frac{9}{2}, \frac{9}{2})$

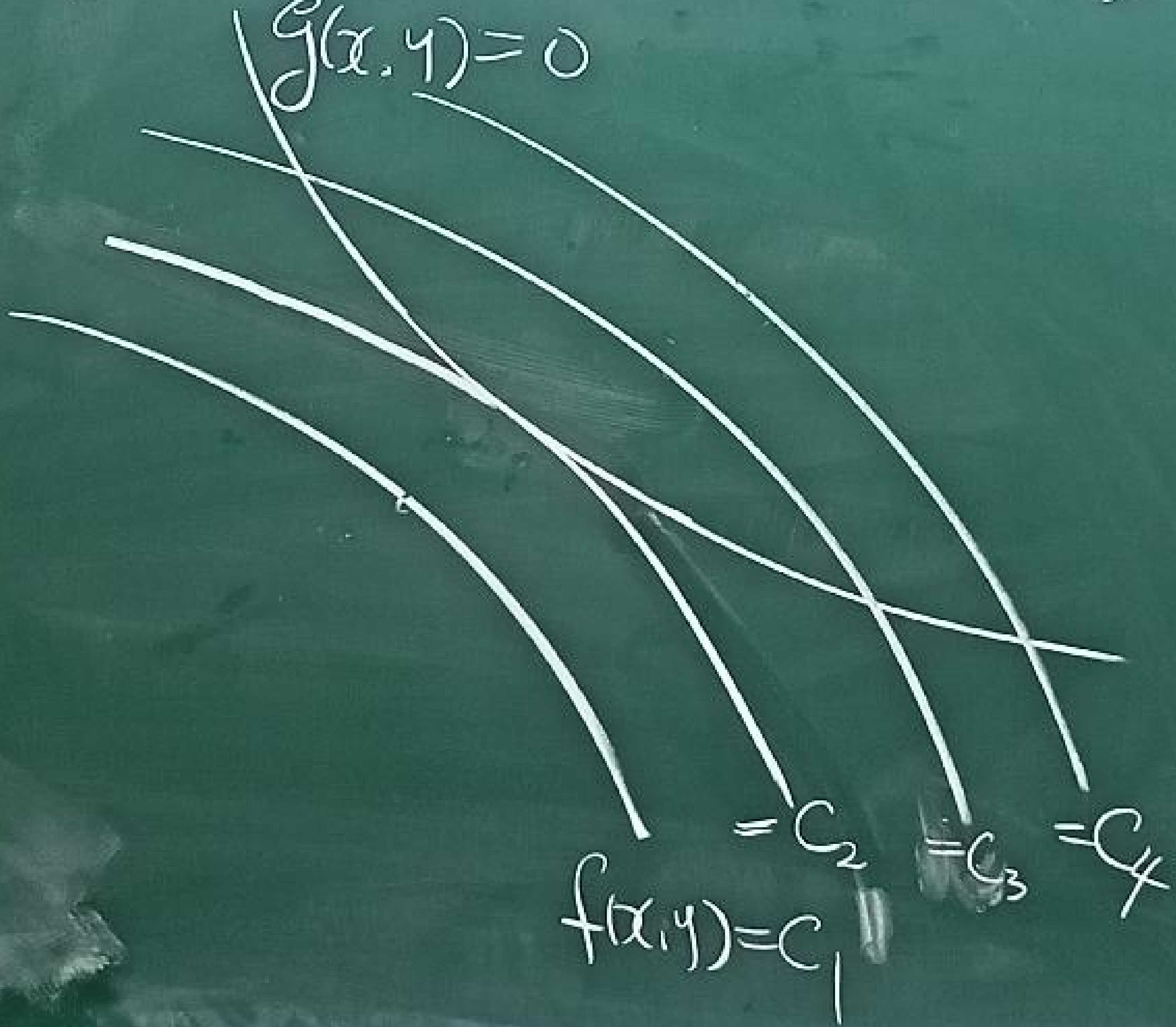
local min $= (9,0).\ (0,9)$

Absolute max $= f(1,1)$

Abs. mini: Compare $f(0,0)$, $f(0,9)$, $f(9,0)$

Constrained optimization
(Lagrangian Multiplier)

<u>Goal</u> Find local min/max of $f(x,y)$ on $\{g(x,y)=0\}$



Local extreme at $(x_0, y_0)$

$$\Rightarrow \begin{cases} g(x,y)=0 \\ f(x,y)=c \end{cases} \text{ are tangent at } (x_0, y_0)$$

$$\Rightarrow \nabla f(x_0, y_0) \parallel \nabla g(x_0, y_0)$$

$$\Rightarrow \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

for some constant $\lambda$

$\Rightarrow$ Solve $(x_0, y_0), \lambda$

from $\begin{cases} g(x_0, y_0) = 0 \\ \nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \end{cases}$

3 equations, 3 unknowns

Eg 1: Find the nearest point to origin on $(x-\frac{1}{2})^2+\frac{y^2}{4}=1$

<u>Sol</u>: Minimize $f(x,y)=(x-0)^2+(y-0)^2$

subject to $g(x,y)=(x-\frac{1}{2})^2+\frac{y^2}{4}-1=0$

$$
\begin{array}{ll}
(1) & \left\{\begin{array}{ll} (x_0-\frac{1}{2})^2+\frac{y_0^2}{4}=1 & (g=0) \\ 2x_0=\lambda(x_0-\frac{1}{2}) & (f_x=\lambda g_x) \\ 2y_0=\lambda\cdot\frac{y_0}{2} & (f_y=\lambda g_y) \end{array}\right. \\
(2) & \\
(3) &
\end{array}
$$

$(3) \Rightarrow y_0=0$ or $\lambda=4$

<u>Case A</u>: $y_0=0$

$$
\left\{\begin{array}{ll} (x_0-\frac{1}{2})^2=1 & (A_1) \\ 2x_0=2\lambda(x_0-\frac{1}{2}) & (A_2) \end{array}\right.
$$

$(A_1) \Rightarrow x_0 = \frac{-1}{2}$ or $\frac{3}{2}$

"+"$(A_2) \Rightarrow \lambda = \frac{1}{2}$, $3$

<u>Case B</u> $\lambda = 4$

"+"$(2) \Rightarrow x_0 = \frac{2}{3}$

"+"$(1) \Rightarrow y_0 = \frac{\pm\sqrt{35}}{3}$

Overall, we have

nearest

$$(x_0, y_0) = \underline{\left(\frac{-1}{2}, 0\right)}, \left(\frac{3}{2}, 0\right), \left(\frac{2}{3}, \frac{\sqrt{35}}{3}\right), \left(\frac{2}{3}, \frac{-\sqrt{35}}{3}\right)$$

$f(x_0, y_0)$ (= distance$^2$)

$$= \underbrace{\frac{1}{4}}_{\text{abs min}}, \frac{9}{4}, \underbrace{\frac{13}{3}, \frac{13}{3}}_{\text{abs max}}$$