Extreme Values and Saddle points

How to find local min/max of $f(x,y)$
(assuming all partial derivatives are cont.)

(1) First derivative test

f has local min/max at (x_0, y_0)

$$\overset{\Rightarrow}{\not\Leftarrow} \quad \nabla f(x_0, y_0) = (0,0)$$

$$\because \Delta z = f(x,y) - f(x_0,y_0) = \overbrace{\nabla f(x_0,y_0)\cdot(\Delta x, \Delta y)}^{\text{leading term}} + \underbrace{(\varepsilon_1, \varepsilon_2)\cdot(\Delta x, \Delta y)}_{\text{smaller}}$$

If $\nabla f(x_0, y_0) \neq (0,0)$

then $\Delta z \begin{matrix} >0 \\ <0 \end{matrix}$ if $(\Delta x, \Delta y) \parallel \begin{matrix} \nabla f / |\nabla f| \\ (-\nabla f / |\nabla f|) \end{matrix}$

<u>Def</u>: (x_0, y_0) is an (interior) critical point of f
if $\nabla f(x_0, y_0) = (0,0)$ or does not exist

It can be shown (P883, n=2) that

$$\Delta f = \nabla f(x_0,y_0)\cdot(\Delta x,\Delta y) \quad \cdots \Delta_1$$

$$+\frac{1}{2}\left(f_{xx}(x_0,y_0)(\Delta x)^2+2f_{xy}(x_0,y_0)\Delta x\Delta y + f_{yy}(x_0,y_0)(\Delta y)^2\right) \cdots \Delta_2$$

P464

$$+O\left((\Delta x)^3,(\Delta x)^2\Delta y,\Delta x(\Delta y)^2,(\Delta y)^3\right) \cdots \Delta_3$$

$\Delta_{1,2,3}$ = (linear, quadratic, cubic) in Δx, Δy

$$|\Delta_3| \leqslant C_1\left(|\Delta x|^3+|\Delta x|^2|\Delta y|+|\Delta x||\Delta y|^2+|\Delta y|^3\right)$$

If $\nabla f(x_0,y_0)=0$, $\Delta_1=0$

$\Delta_2 = A(\Delta x)^2+2B\Delta x\Delta y+C(\Delta y)^2$ = leading term

$A=f_{xx}(x_0,y_0)$, $B=f_{xy}(x_0,y_0)$, $C=f_{yy}(x_0,y_0)$

$\Delta_3 \ll \Delta_2$ $\left(\lim \frac{\Delta_3}{\Delta_2}=0\right)$ if $\Delta_2\neq 0$

leading term $= \Delta_2$

$$= A(\Delta x)^2 + 2B\Delta x\Delta y + C(\Delta y)^2 \cdots (*)$$

$$= A\left(\left(\Delta x + \frac{B}{A}\Delta y\right)^2 - \frac{B^2 - AC}{A^2}(\Delta y)^2\right)$$

$$= \begin{cases} A\cdot(\text{sum of squares}) & \text{if } B^2 - AC < 0 \\ A\cdot(\text{difference of squares}) & \text{if } B^2 - AC > 0 \end{cases}$$

If $(\Delta x, \Delta y) \neq (0,0)$

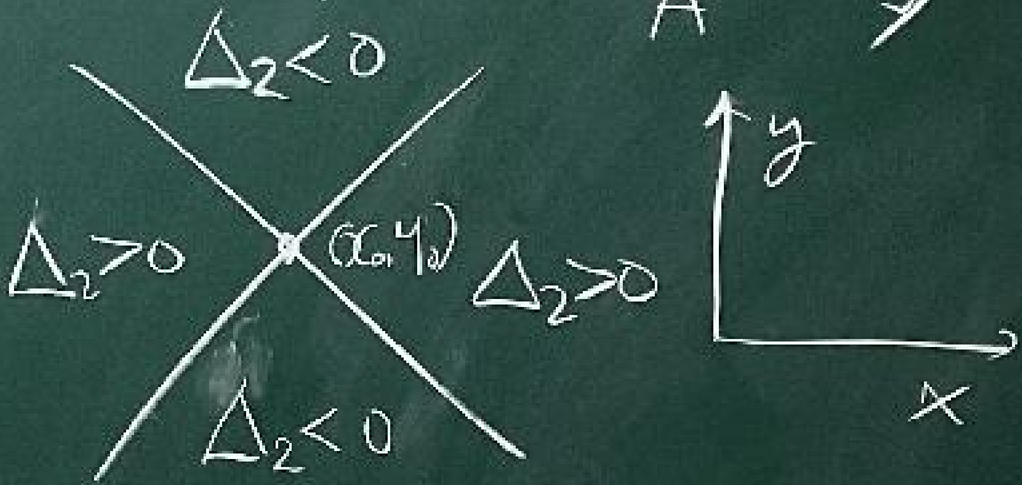
$$\begin{cases} \Delta_2 \text{ always} > 0 \iff \begin{cases} A > 0 \\ B^2 - AC < 0 \end{cases} \\ \Delta_2 \text{ always} < 0 \iff \begin{cases} A < 0 \\ B^2 - AC < 0 \end{cases} \\ \Delta_2 \text{ can change sign} \iff B^2 - AC > 0 \end{cases}$$

The Second Derivative Test: $(f_x = 0,\ f_y = 0)$

$$\begin{cases} f_{xx} > 0,\ f_{xy}^2 - f_{xx}f_{yy} < 0 : \text{local min} \\ f_{xx} < 0,\ f_{xy}^2 - f_{xx}f_{yy} < 0 : \text{local max} \\ \qquad\quad f_{xy}^2 - f_{xx}f_{yy} > 0 : \text{saddle point} \end{cases}$$

If $B^2 - AC > 0$

$$\Delta_2 = A\cdot\left(\Delta x - \frac{-B+\sqrt{B^2-AC}}{A}\Delta y\right)\cdot\left(\Delta x - \frac{-B-\sqrt{B^2-AC}}{A}\Delta y\right)$$



Def: (x_0, y_0) is a saddle point of f

if $\begin{cases} \nabla f = (0,0) \\ f_{xy}^2 - f_{xx} f_{yy} > 0 \end{cases}$ at (x_0, y_0)

Summary: How to find interior local extremes of f. assuming all partial derivatives of f are cont.)

Step 1: Find all critical points (x_0, y_0)

where $\nabla f(x_0, y_0)$ where $\nabla f(x_0, y_0) = (0,0)$

Step 2: Evalute $D = (f_{xy}^2 - f_{xx} f_{yy})(x_0, y_0)$

and $A = f_{xx}(x_0, y_0)$

(i) If $A < 0$, $D < 0 \Rightarrow$ local max

(ii) If $A > 0$, $D < 0 \Rightarrow$ local min

(iii) If $D > 0 \Rightarrow$ Saddle point (not local max, not local min)

(iv) If $D = 0 \Rightarrow$ inconclusive

Eg1. find local extremes of $f(x,y) = x^2 + y^2 - 4y + 9$

Sol: Method 1

$f_x = 2x$, $f_y = 2(y-2)$

$\Rightarrow$ critical point $= (0,2)$ only

$f_{xx}(0,2) = 2$, $f_{xy}(0,2) = 0$

$f_{yy}(0,2) = 2 \Rightarrow D = -4$

$A > 0$, $D < 0$

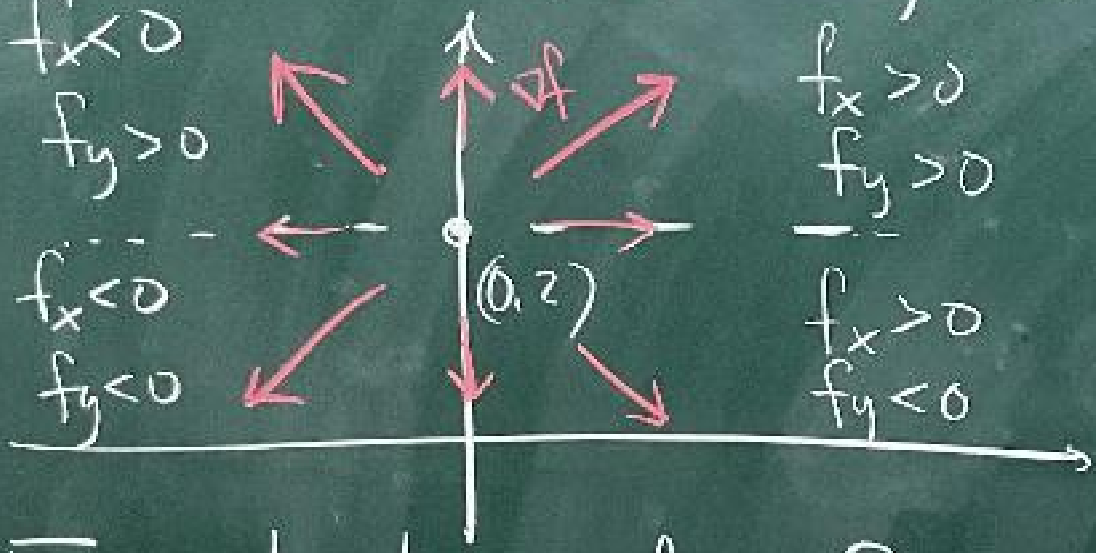
$\Rightarrow$ local min $= f(0,2) = 5$

local max: none

Method 2: Gradient Analysis

$$f_x = 2x \begin{cases} >0 & \text{if } x>0 \\ <0 & \text{if } x<0 \end{cases}$$

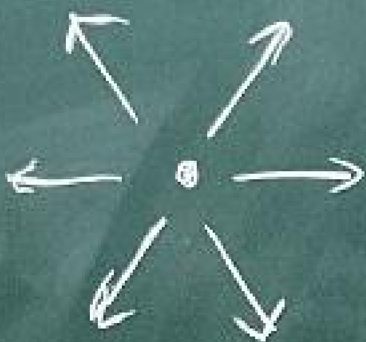
$$f_y = 2(y-2) \begin{cases} >0 & y>2 \\ <0 & y<2 \end{cases}$$



From directions of ∇f near $(x_0,y_0)=(0,2)$

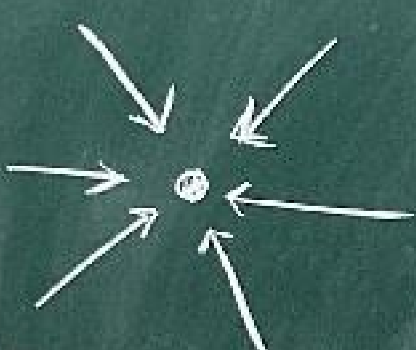
$\Rightarrow f(0,2)$ is a local min

Remark: Near a critical point where $\nabla f(x_0, y_0) = (0,0)$



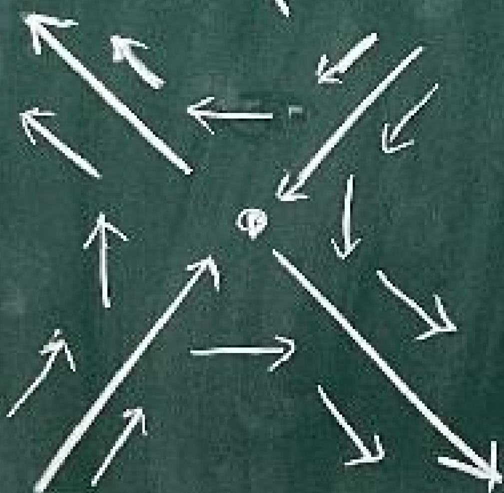
∇f points outward

$\Rightarrow$ local min



∇f points inward

$\Rightarrow$ local max



∇f points inward in some directions and outward in some directions $\Rightarrow$ Not local min, Not local max

(e.g. saddle points)