

Ex 2. Find the tangent plane and normal line of $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$ at $(1, 2, 3)$.

Sol tangent plane $\perp \nabla F(1, 2, 3)$
normal line $\parallel \nabla F(1, 2, 3)$

$$F(x, y, z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9}$$

$$\text{Surface: } F(x, y, z) = 3$$

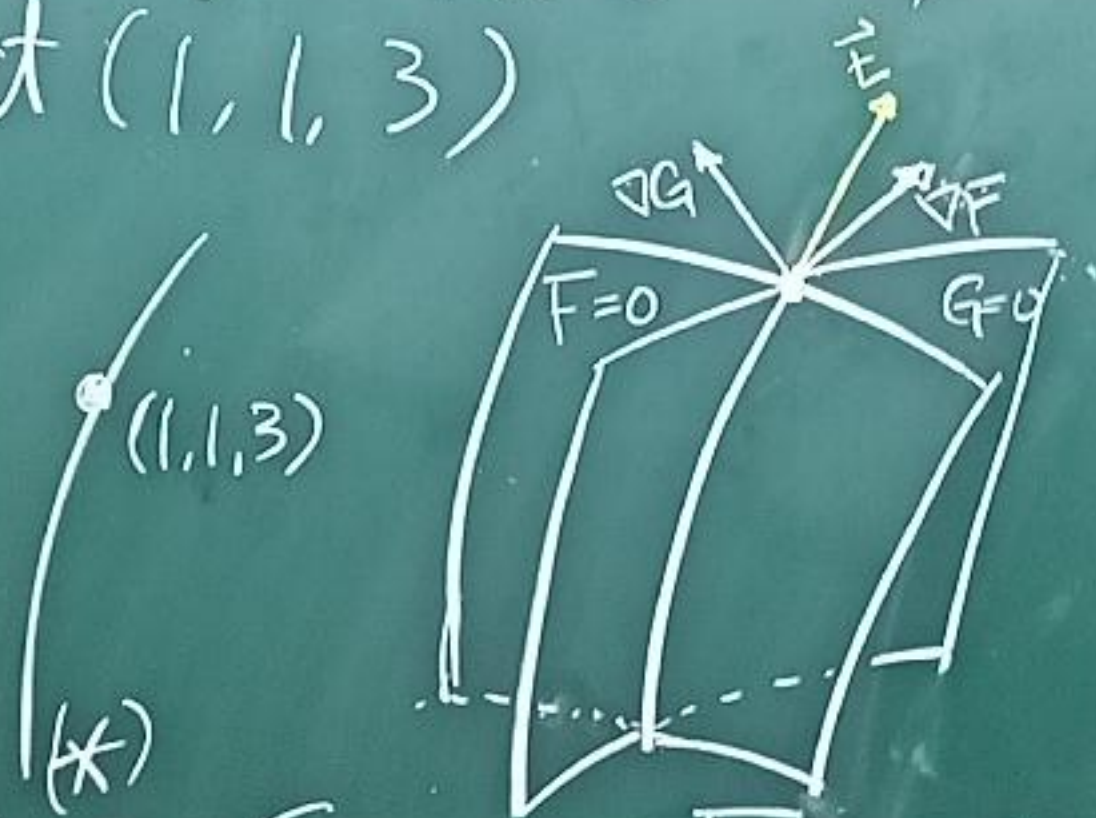
$$\nabla F = (F_x, F_y, F_z) = \left(2x, \frac{y}{2}, \frac{2z}{9}\right)$$

$$\nabla F(1, 2, 3) = \left(2, 1, \frac{2}{3}\right)$$

$$\text{tangent plane: } (x-1, y-2, z-3) \cdot \left(2, 1, \frac{2}{3}\right) = 0$$

$$\text{normal line } \begin{cases} x(t) = 1 + 2t \\ y(t) = 2 + t \\ z(t) = 3 + \frac{2}{3}t \end{cases}$$

Eg 3 Find the tangent
 line of $\begin{cases} x^2 + y^2 - 2 = 0 \\ x + z = 4 \end{cases} (*)$
 at $(1, 1, 3)$



If $C = \left\{ (x, y, z) \mid \begin{array}{l} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{array} \right\}$
 and $(x_0, y_0, z_0) \in C$

\vec{T} : tangent vector of C
 at (x_0, y_0, z_0) . Here $F(x, y, z) = x^2 + y^2 - 2$
 $G(x, y, z) = x + z - 4$

$$C \subset S_1 =$$

Since $C \subset S_1 = \left\{ (x, y, z), F(x, y, z) = \text{const} \right\}$
 $= F(x_0, y_0, z_0)$

$\Rightarrow \vec{T}$ is a tangent vector of S_1
at (x_0, y_0, z_0)

$$\Rightarrow \vec{T} \perp \nabla F(x_0, y_0, z_0)$$

Similarly, $\vec{T} \perp \nabla G(x_0, y_0, z_0)$

$$\therefore \vec{T} \parallel \nabla F(x_0, y_0, z_0) \times \nabla G(x_0, y_0, z_0)$$

$$= (2x_0, 2y_0, 0) \times (1, 0, 1)$$

$$= (2, 2, 0) \times (1, 0, 1)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (2, -2, -2)$$

Ans $\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z-3}{-2}$

or $x(t) = 1 + 2t, y(t) = 2 - 2t, z(t) = 3 - 2t$

Linearization

If $f(x, y)$ is diff. at (x_0, y_0)

$$\Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - L(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

$\therefore L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
is a good approximation of $z = f(x, y)$
near $(x_0, y_0, f(x_0, y_0))$

Linearization: use $L(x, y)$ to approx. $f(x, y)$

$$\Leftrightarrow \Delta z \approx f_x(x_0, y_0) \Delta x + f_y(x_0, y_0) \Delta y \quad (\star)$$

($df = f_x(x_0, y_0) dx + f_y(x_0, y_0) dy$)

Note: If "=" holds in (\star)

$$\Rightarrow z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\Rightarrow z = L(x, y)$$

Eg 1: $f(x, y) = x^2 - xy + \frac{y^2}{2} + 3$

Find approx. value of $f(3.1, 1.9)$

Sol: take $(x_0, y_0) = (3, 2)$

$$f(3, 2) = 8, \quad f_x(3, 2) = 4$$

$$f_y(3, 2) = -1$$

App. value = $L(3.1, 1.9)$

$$= f(3, 2) + f_x(3, 2) \cdot (3.1 - 3) + f_y(3, 2) \cdot (1.9 - 2)$$

$$= 8 + 4 \cdot 0.1 + (-1) \cdot (-0.1) = 8.5$$

Remark: The same as

$$\Delta z = z - f(3, 2) \approx f_x(3, 2) \Delta x + f_y(3, 2) \Delta y$$

$$\Rightarrow z \approx 8.5$$

Error of linearization

$$f(x, y) \cong L(x, y)$$

How large is $|f(x, y) - L(x, y)|$?

1D case ($y = f(x)$)

$$f(x) = \underbrace{f(x_0) + f'(x_0)(x - x_0)}_{L(x)} + \underbrace{\frac{f''(\xi)}{2}(x - x_0)^2}_{\text{error}}$$

(Taylor's formula ($n=1$) on P 883)

2D case ($z = f(x, y)$)

$$f(x, y) = \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x, y)}$$

$$+ \frac{1}{2} \left(f_{xx}(\xi, \eta)(x - x_0)^2 + 2f_{xy}(\xi, \eta)(x - x_0)(y - y_0) + f_{yy}(\xi, \eta)(y - y_0)^2 \right)$$

where (ξ, η) lies on the segment (x_0, y_0) (x, y)

error

Corollary: If $f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ are all continuous in an open region R , $(x_0, y_0) \in R$

and $|f_{xx}|, |f_{xy}|, |f_{yy}| \leq M$ on R

Then $|f(x, y) - L(x, y)|$

$$\leq \frac{1}{2} (M|x-x_0|^2 + 2M|x-x_0||y-y_0| + M|y-y_0|^2)$$

$$= \frac{M}{2} (|x-x_0| + |y-y_0|)^2$$

Error estimate for linear approximation

In Eg 1, $f_{xx} = 2, f_{xy} = -1, f_{yy} = 1$ ($\therefore M = 2$)

$$\therefore |f(3.1, 1.9) - 8.5| \leq \frac{M}{2} (0.1 + 0.1)^2 = 0.04$$

Eg2: Volume of Cylinder

$$V(r, h) = \pi r^2 h$$

If $r_0 = 1, h_0 = 5$

$$r_1 = r_0 + \Delta r, \quad \Delta r = 0.1$$

$$h_1 = h_0 + \Delta h, \quad \Delta h = -0.1$$

∴ |Error|

$$\leq \frac{10\pi}{2} (0.1 + 0.1)^2$$

$$= 0.2\pi$$

$$\Delta V = V(r_1, h_1) - V(r_0, h_0) = ?$$

Ans: $\Delta V \approx V_r(r_0, h_0) \Delta r + V_h(r_0, h_0) \Delta h$

$$= 2\pi r_0 h_0 \Delta r + \pi r_0^2 \Delta h = 0.9\pi$$

$$|\text{Error}| \leq \frac{M}{2} (|\Delta r| + |\Delta h|)^2$$

Find M , such that $|V_{rr}|, |V_{rh}|, |V_{hh}| \leq M$

$$V_{rr}(C_1, C_2) = 2\pi C_2 \quad (r_0, h_0) = (1, 5)$$

$$V_{rh}(C_1, C_2) = 2\pi C_1$$

$$|C_1| \leq 1.1, |C_2| \leq 5 \Rightarrow M = 10\pi \text{ will do}$$

