Eg3 $f(x,y) = \begin{cases} \dfrac{4xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Evaluate $D_{\vec{u}} f(0,0)$ and $\nabla f(0,0) \cdot \vec{u}$

Ans: $f_x(0,0) = \lim\limits_{x \to 0} \dfrac{f(x,0) - f(0,0)}{x - 0} = 0$

$f_y(0,0) = \lim\limits_{x \to 0} \dfrac{f(0,y) - f(0,0)}{y - 0} = 0$

$D_{\vec{u}} f(0,0) = \lim\limits_{s \to 0} \dfrac{f(su_1, su_2) - f(0,0)}{s - 0} \quad \left(\underline{\underline{u_1^2 + u_2^2 = 1}} \right)$

$= \lim\limits_{s \to 0} \dfrac{\frac{4s^3 u_1 u_2}{s^2}}{s - 0} = 4u_1u_2 \neq \underbrace{\nabla f(0,0)}_{(0,0)} \cdot \vec{u}$

Eg 4 $f(x,y) = ax + by + c$

$\nabla f(0,0)\vec{u} \overset{?}{=} D_{\vec{u}} f(0,0)$

Sol $f_x(0,0) = a, \quad f_y(0,0) = b$

$$D_{\vec{u}} f(0,0) = \lim_{s \to 0} \frac{(asu_1 + bsu_2 + c) - c}{s}$$

$$= au_1 + bu_2 = \nabla f(0,0) \cdot \vec{u}$$

Remark. f is diff. at (x_0, y_0)

$\Rightarrow$ $z = f(x,y)$ has a tangent plane at $(x_0, y_0, f(x_0, y_0))$

$\Rightarrow$ f is close to a linear function

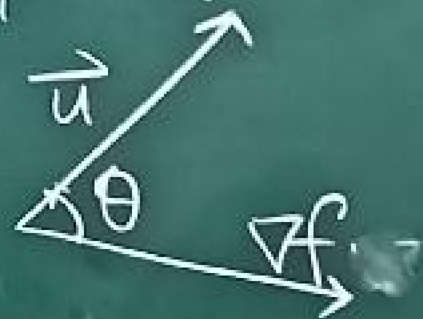
$\Rightarrow$ $D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

Properties of ∇f

If f is diff. at (x_0, y_0)

$\Rightarrow D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$

$= |\nabla f(x_0, y_0)| \cdot |\vec{u}| \cos\theta$

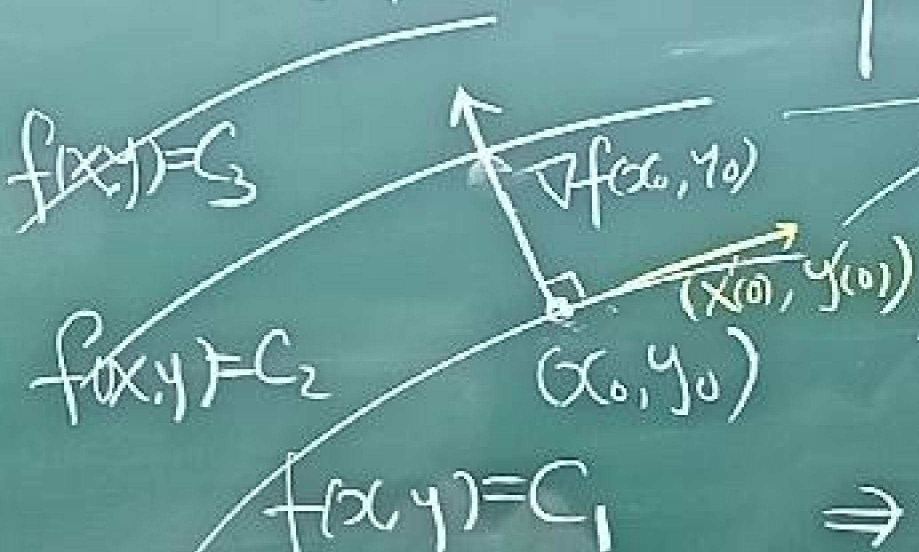


$\Rightarrow$ (1) f increase (decrease) most rapidly in the direction of $\vec{u}$ if $\cos\theta = 1$ (-1)

i.e. in the direction of ∇f $(-\nabla f)$

(2) $D_{\vec{u}} f(x_0, y_0) = 0 \Leftrightarrow \nabla f(x_0, y_0) \perp \vec{u}$

(3) Another interpretation of ∇f



level curves of f $=\{(x,y) \mid f(x,y)=c\}$

$\Rightarrow \nabla f \perp$ level curve.

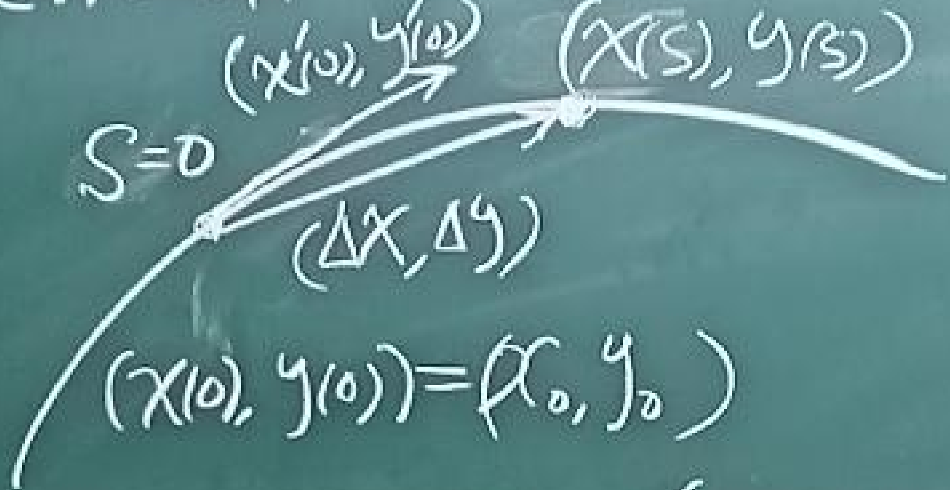
pf: Let $(x(s),y(s))$ be a level curve of f (differentiable) with $(x(0),y(0))=(x_0,y_0)$

$\Rightarrow f(x(s),y(s))=\text{constant}$

$\frac{d}{ds}\Big|_{s=0} \Rightarrow \nabla f(x_0,y_0)\cdot(x'(0),y'(0))=0$

$\vec{u}$ $(x'(0),y'(0))$ = a tangent vector of level curve

Remark



$$\begin{pmatrix} x'(0) \\ y'(0) \end{pmatrix} = \lim_{s \to 0} \begin{pmatrix} \dfrac{x(s)-x(0)}{s-0} \\ \dfrac{y(s)-y(0)}{s-0} \end{pmatrix}$$

$$= \lim_{\Delta s \to 0} \frac{1}{\Delta s} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad \Delta s = s - 0$$

$=$ a tangent vector

$\therefore (x'(0), y'(0))$ points in the direction of tangent line at $(x(0), y(0))$

Eg1 Find the tangent line (normal) of $\frac{x^2}{4}+y^2=2$ at $(-2,1)$

Sol Let $f(x,y)=\frac{x^2}{4}+y^2$

$\therefore \frac{x^2}{4}+y^2=2$ is a level curve of f

$\therefore \nabla f(-2,1)=$ normal vector of the level curve at $(-2,1)$

tangent line:

$$(x+2,\ y-1)\cdot \underbrace{\nabla f(-2,1)}_{\overset{\parallel}{(-1,\,2)}}=0$$

Normal line:

$$\frac{y-1}{x-2}=\frac{2}{-1}$$

Algebraic Rule for gradient

(i) $\nabla(f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla(c f(x,y)) = c \nabla f(x,y)$

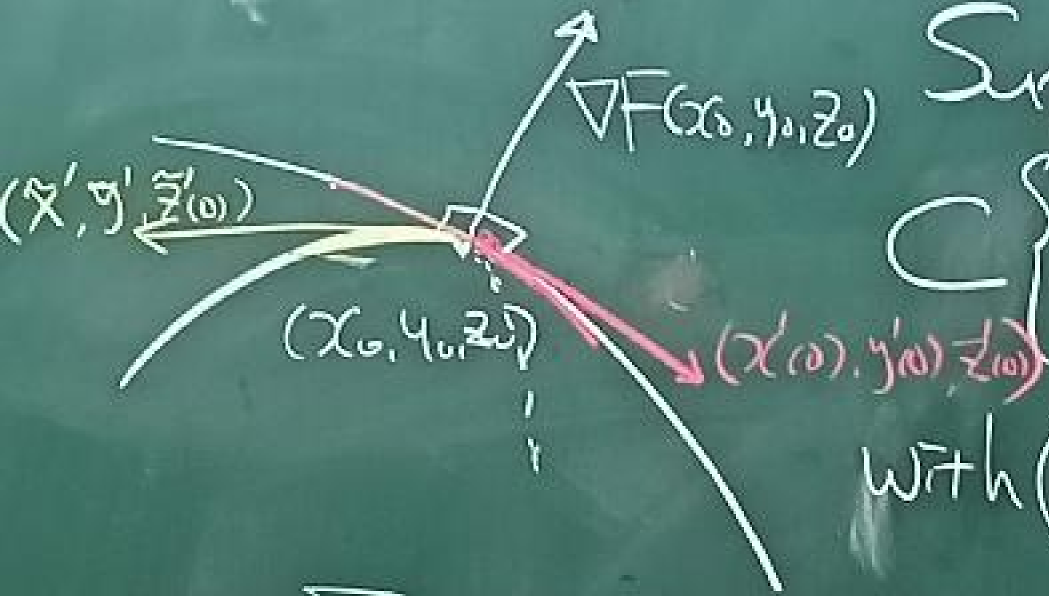
(iii) $\nabla(f \cdot g) = f \nabla g + g \nabla f$

(iv) $\nabla\left(\frac{f}{g}\right) = \frac{g \nabla f - f \nabla g}{g^2}$

pf: check x-component
and y-component directly

Tangent plane and normal line of level surface of $F(x,y,z)$ (diff.)

: $\{(x,y,z) \mid F(x,y,z)=C\}$



Suppose $\{x(t), y(t), z(t)\} \subset \{\text{level surface of } F\}$ at (x_0,y_0,z_0)

with $(x(0),y(0),z(0)) = (x_0,y_0,z_0)$

$$\Rightarrow F(x(t),y(t),z(t)) = \text{constant} = F(x_0,y_0,z_0)$$

$$\left.\frac{d}{dt}\right|_{t=0} \Rightarrow \nabla F(x_0,y_0,z_0)\cdot \underbrace{(x'(0),y'(0),z'(0))}_{\text{a tangent vector of the level surface at }(x_0,y_0,z_0)} = 0$$

For any curve $(x(t), y(t), z(t))$

on the surface passing (x_0, y_0, z_0)

$\nabla F(x_0, y_0, z_0) \perp (x'(0), y'(0), z'(0))$

$\Rightarrow \nabla F(x_0, y_0, z_0) \perp$ any tangent vector at (x_0, y_0, z_0)

$\Rightarrow \nabla F(x_0, y_0, z_0) \perp$ tangent plane at (x_0, y_0, z_0)

tangent plane

$$(x-x_0, y-y_0, z-z_0) \cdot \nabla F(x_0, y_0, z_0) = 0$$

normal line: $\dfrac{x-x_0}{F_x(x_0,y_0,z_0)} = \dfrac{y-y_0}{F_y(x_0,y_0,z_0)} = \dfrac{z-z_0}{F_z(x_0,y_0,z_0)}$

or $\begin{cases} x(t) = x_0 + F_x(x_0,y_0,z_0)\,t \\ y(t) = y_0 + F_y(x_0,y_0,z_0)\,t \\ z(t) = z_0 + F_z(x_0,y_0,z_0)\,t \end{cases}$