# Chain Rule

Single variable function

$$\frac{d}{dx} f(g(x)) = \left.\frac{df}{dy}\right|_{y=g(x)} \cdot \frac{dg(x)}{dx}$$

Two variables function $z = f(x, y)$

$$\frac{d}{dt} f(x(t), y(t)) = \lim_{\substack{t \to t_0}} \frac{\Delta z}{\Delta t}$$

$$\left( \Delta z = f(x(t), y(t)) - f(x(t_0), y(t_0)) \right)$$

$\Delta t = t - t_0$

(assume $f$ is differentiable)

$$= \lim_{\substack{\Delta t \to 0 \\ (t \to t_0)}} \left(f_x(x(t), y(t)) + \varepsilon_1\right)\frac{\Delta x}{\Delta t} + \left(f_y(x(t), y(t)) + \varepsilon_2\right)\frac{\Delta y}{\Delta t}$$

$(*)$

When $\Delta t \to 0$ $(t \to t_0)$

we have $(x(t), y(t)) \Rightarrow (x(t_0), y(t_0))$

(assume $x(t)$, $y(t)$ are also differentiable at $t_0$, therefore continuous at $t_0$)

$\Rightarrow (\Delta x, \Delta y) \to (0,0)$

$\Rightarrow \lim\limits_{\Delta t \to 0} \varepsilon_1 = 0 = \lim\limits_{\Delta t \to 0} \varepsilon_2$

$\therefore (*)$

$$= f_x(x(t_0), y(t_0)) \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$+ f_y(x(t_0), y(t_0)) \lim_{\Delta t} \frac{\Delta y}{\Delta t}$$

$$= f_x(x(t_0), y(t_0))\, x'(t_0) + f_y(x(t_0), y(t_0))\, y'(t_0)$$

In other words

$$\frac{d}{dt} f(x(t), y(t)) = f_x(x(t), y(t))\, x'(t) + f_y(x(t), y(t))\, y'(t)$$

Similarly, $\partial_s f(x(s,t), y(s,t), z(s,t))$

$$\begin{aligned} &= f_x(x(s,t), y(s,t), z(s,t)) \cdot \partial_s x(s,t) \\ &\quad + f_y(\qquad) \cdot \partial_s y(s,t) \\ &\quad + f_z(\qquad)\, \partial_s z(s,t) \end{aligned}$$

$\partial_t f(x(s,t), y(s,t), z(s,t))$

$$\begin{aligned} &= f_x(\;\cdots\;) \cdot \partial_t x(s,t) \\ &\quad + f_y(\;\cdots\;)\, \partial_t y(s,t) \\ &\quad + f_z(\qquad)\, \partial_t z(s,t) \end{aligned}$$

# Implicit differentiation revisited

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $F(x,y,z)=0$

(i.e. If $z(x,y)$ is implicitly defined by $F(x,y,z)=0$, find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$)

<u>Sol</u> $F(x,y,z(x,y)) = 0$ (as two functions of $(x,y)$)

$$\Rightarrow \partial_x F(x,y,z(x,y)) = 0$$

$$\Rightarrow \partial_1 F \cdot \underbrace{\frac{\partial x}{\partial x}}_{=1} + \partial_2 F \cdot \underbrace{\frac{\partial y}{\partial x}}_{=0} + \partial_3 F \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \partial_1 F(x,y,z(x,y)) + \partial_3 F(x,y,z(x,y)) \cdot \frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} \text{ at } (x,y,z) = -\frac{\partial_1 F(x,y,z)}{\partial_3 F(x,y,z)} = -\frac{\partial_x F(x,y,z)}{\partial_z F(x,y,z)}$$

Similarly

$$\frac{\partial z}{\partial y} \text{ at } (x,y,z) = -\frac{\partial_y F(x,y,z)}{\partial_z F(x,y,z)}.$$

Eg 1 Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ at $(1,1,1)$

from $F(x,y,z) = xy + z^3x - 2yz = 0$

Ans. $\partial_x F(1,1,1) = y + z^3 \big|_{(1,1,1)} = 2$

$\partial_y F(1,1,1) = x - 2z \big|_{(1,1,1)} = -1$

$\partial_z F(1,1,1) = 3z^2x - 2y \big|_{(1,1,1)} = 1$

$\therefore \frac{\partial z}{\partial x} \text{ at } (1,1,1) = -\frac{2}{1} = -2$

$\frac{\partial z}{\partial y} \text{ at } (1,1,1) = -\frac{(-1)}{1} = 1$

# Directional derivative

Def: $\left(\dfrac{df}{ds}\right)_{\vec{u},P_0} \left(= D_{\vec{u}} f(x_0,y_0)\right)$

$(P_0=(x_0,y_0))$

(Derivative of $f(x,y)$ at $P_0=(x_0,y_0)$ in the direction of the unit vector $\vec{u}=(u_1,u_2)$
$(u_1^2+u_2^2=1)$)

$$\overset{\text{def}}{=\!=} \lim_{s\to 0} \frac{f(x_0+su_1,\, y_0+su_2)-f(x_0,y_0)}{s-0}$$

$$\left(= \lim_{s\to 0} \frac{f(x(s),y(s))-f(x(0),y(0))}{s-0}\right.$$

where $x(s)=x_0+su_1$, $y(s)=y_0+su_2$ )

Thm If $f(x,y)$ is differentiable at $(x_0,y_0)$, then $D_{\vec{u}} f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\vec{u}$

where $\nabla f = (f_x, f_y)$ (gradient of $f$)

pf: $\Delta z = (f_x(x_0,y_0)+\varepsilon_1)\Delta x + (f_y(x_0,y_0)+\varepsilon_2)\Delta y$

Let $x(s) = x_0 + su_1$, $y(s) = y_0 + su_2$

$$\Delta x = x(s) - x(0) = su_1$$
$$\Delta y = y(s) - y(0) = su_2$$

$\Delta s = s - 0$

$$\Rightarrow D_{\vec{u}} f(x_0,y_0) = \lim_{\Delta s\to 0}\frac{\Delta z}{\Delta s}$$
$$= \lim_{\Delta s\to 0}\left((f_x(x_0,y_0)+0)\frac{\Delta x}{\Delta s} + (f_y(x_0,y_0)+0)\frac{\Delta y}{\Delta s}\right)$$
$$= f_x(x_0,y_0)u_1 + f_y(x_0,y_0)u_2 = \nabla f(x_0,y_0)\cdot\vec{u}$$

Eg 2 $f(x,y) = x^2 + xy$

Note: f is differentiable

let $\vec{u} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$D_{\vec{u}} f(1,2) = ?$

Sol $= \nabla f(1,2) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$= \frac{f_x(1,2)}{\sqrt{2}} + \frac{f_y(1,2)}{\sqrt{2}}$

$f_x = 2x + y, \quad f_y = x$

Ans $= \frac{4+1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$