In definition of $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$

$$L(x,y) \overset{\text{def}}{=} f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f(x,y) = L(x,y) + \varepsilon_1(x-x_0) + \varepsilon_2(y-y_0) \quad (1)$$

$$\Leftrightarrow f(x,y) = L(x,y) + \varepsilon\cdot\sqrt{(x-x_0)^2+(y-y_0)^2} \quad (2)$$

$$\Leftrightarrow \Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \quad (3)$$

$$\Leftrightarrow \Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon\cdot\sqrt{(\Delta x)^2+(\Delta y)^2} \quad (4)$$

where $\Delta z = f(x,y) - f(x_0,y_0)$, $\Delta x = x - x_0$, $\Delta y = y - y_0$

and $\lim_{(x,y)\to(x_0,y_0)} (\varepsilon_1, \varepsilon_2, \varepsilon) = (0,0,0)$

(1)-(4) are all the same

The textbook uses (3)

Def

$z = f(x,y)$
$z = g(x,y)$ are tangent at (x_0, y_0, z_0)

if (i) $f(x_0, y_0) = g(x_0, y_0) = z_0$

(ii) $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - g(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

$\therefore$ f is differetiable at (x_0, y_0)

$\Longleftrightarrow$ $z = f(x,y)$ and $z = L(x,y)$ are tangent at $(x_0, y_0, f(x_0, y_0))$

In fact, it can be shown that, if $z = f(x, y)$ has a tangent plane at (x_0, y_0, z_0), then $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ must exist, and the tangent plane must be

$$z = L(x, y)$$

(See supplement)

Thm 2: If $f, f_x, f_y, f_{xy}, f_{yx}$ are all cont. in an open region R and $(x_0, y_0) \in R$

Then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$

$$= \partial_y \partial_x f(x_0, y_0) \qquad \overset{\|}{\partial_x \partial_y f(x_0, y_0)}$$

(R is an open region if R has no boundary point)

Note:

$f_{xy}(x_0, y_0), f_{yx}(x_0, y_0)$ both exist

$\not\Rightarrow f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$

Eg1 $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$

Then $f_{xy}=f_{yx}$ on $\mathbb{R}^2\setminus(0,0)$ (direct computation)

How about $f_{xy}(0,0) \overset{?}{=} f_{yx}(0,0)$

<u>Sol</u> $f_{xy}(0,0)=\lim\limits_{y\to 0}\dfrac{f_x(0,y)-f_x(0,0)}{y-0}$

$f_x(0,y)=\partial_x\left(xy\dfrac{x^2-y^2}{x^2+y^2}\right)\Big|_{(0,y)}=\cdots$

$f_x(0,0)=\lim\limits_{x\to 0}\dfrac{f(x,0)-f(0,0)}{x-0}=\lim\limits_{x\to 0}\dfrac{x\cdot 0\,\frac{x^2-0}{x^2+0}-0}{x}=0$

Similarly for $f_{yx}(0,0)$ (homework)

Thm 3: R is an open region $(x_0, y_0) \in R$. If f, f_x, f_y are all defined in R and continuous at (x_0, y_0), then f is differentiable at (x_0, y_0).

Pf: See Appendix 9.

Eg2 $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} , & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Is $f(x,y)$ cont at $(0,0)$?

differentiable at $(0,0)$?

Ans (i) $\lim\limits_{(x,y)\to(0,0)} f(x,y) \overset{?}{=} 0$ (yes)

(ii) Step 1. find $L(x,y)$

$f_x(0,0) = 0 = f_y(0,0) \Rightarrow L(x,y) = 0$

(exercise)

Step 2 $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y) - 0}{\sqrt{x^2+y^2}} \overset{?}{=} 0$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist Ans. NO

Thm 4 $f(x,y)$ is diff. at (x_0,y_0) (1)

$\Rightarrow f(x,y)$ is cont. at (x_0,y_0) (2)

pf: (1) $\Leftrightarrow \Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$

$= (f_x(x_0,y_0)+\varepsilon_1)\Delta x + (f_y(x_0,y_0)+\varepsilon_2)\Delta y$

$$\Rightarrow \lim_{(\Delta x,\Delta y)\to(0,0)} \Delta z = 0$$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) - f(x_0,y_0) = 0$$

$$\Rightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0) \Leftrightarrow (2)$$