

In definition of $\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y)$

$$L(x,y) \stackrel{\text{def}}{=} f(x_0,y_0) + f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)$$

$$f(x,y) = L(x,y) + \varepsilon_1(x-x_0) + \varepsilon_2(y-y_0)$$

$$\Leftrightarrow f(x,y) = L(x,y) + \varepsilon \cdot \sqrt{(x-x_0)^2 + (y-y_0)^2}$$

$$\Leftrightarrow \Delta z = f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$$

$$\Leftrightarrow \Delta z = f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y + \varepsilon \cdot \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

where $\Delta z = f(x,y) - f(x_0,y_0)$, $\Delta x = x - x_0$, $\Delta y = y - y_0$

and $\lim_{(x,y) \rightarrow (x_0,y_0)} (\varepsilon_1, \varepsilon_2, \varepsilon) = (0, 0, 0)$

Def.

$z = f(x, y)$ and $z = g(x, y)$ are tangent at (x_0, y_0, z_0)

if (i) $f(x_0, y_0) = g(x_0, y_0) = z_0$

$$(ii) \lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - g(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

$\therefore f$ is differentiable at (x_0, y_0)

$\Leftrightarrow z = f(x, y)$ and $z = L(x, y)$ are tangent at $(x_0, y_0, f(x_0, y_0))$

In fact, it can be shown
that, if $z = f(x, y)$ has a
tangent plane at (x_0, y_0, z_0) ,
then $f_x(x_0, y_0)$, $f_y(x_0, y_0)$
must exist, and the
tangent plane must be

$$z = L(x, y)$$

(See Supplement)

Thm 2: If $f, f_x, f_y, f_{xy}, f_{yx}$
are all cont. in an open region R
and $(x_0, y_0) \in R$

$$\text{Then } f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0) \\ = \partial_y \partial_x f(x_0, y_0) \quad \partial_x \partial_y f(x_0, y_0)$$

(R is an open region if
 R has no boundary point)

Note:

$f_{xy}(x_0, y_0), f_{yx}(x_0, y_0)$ both exist

$$\Rightarrow f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$$

$$\text{Eg 1 } f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Then $f_{xy} = f_{yx}$ on $\mathbb{R}^2 - (0, 0)$ (direct computation)

How about $f_{xy}(0, 0) \stackrel{?}{=} f_{yx}(0, 0)$

Sol $f_{xy}(0, 0) = \lim_{y \rightarrow 0} \frac{f_x(0, y) - f_x(0, 0)}{y - 0}$

$$f_x(0, y) = \left. \partial_x \left(xy \frac{x^2 - y^2}{x^2 + y^2} \right) \right|_{(0, y)} = \dots$$

$$f_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x \cdot 0 \frac{x^2 - 0}{x^2 + 0} - 0}{x} = 0$$

Similarly for $f_{yx}(0, 0)$ (homework)

Thm 3 : R is an open region
 $(x_0, y_0) \in R$. If f, f_x, f_y
are all defined in R
and continuous at (x_0, y_0) ,
then f is differentiable
at (x_0, y_0) .

Pf : See Appendix 9.

Ex 2 $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Is $f(x,y)$ cont at $(0,0)$?
differentiable at $(0,0)$?

Ans (i) $\lim_{(x,y) \rightarrow (0,0)} f(x,y) \neq 0$ (yes)

(ii) Step 1: find $L(x,y)$

$f_x(0,0) = 0 = f_y(0,0) \Rightarrow L(x,y) = 0$
(exercise)

Step 2 $\lim_{(x,y) \rightarrow (0,0)} \frac{f(x,y) - 0}{\sqrt{x^2+y^2}} \neq 0$

$= \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$ does not exist
Ans. NO

Thm 4 $f(x, y)$ is diff. at (x_0, y_0) (1)
 $\Rightarrow f(x, y)$ is cont. at (x_0, y_0) (2)

pf. (1) $\Leftrightarrow \Delta z = f'_x(x_0, y_0) \Delta x + f'_y(x_0, y_0) \Delta y$
 $+ \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$
 $= (f'_x(x_0, y_0) + \varepsilon_1) \Delta x + (f'_y(x_0, y_0) + \varepsilon_2) \Delta y$

$$\Rightarrow \lim_{(\Delta x, \Delta y) \rightarrow (0, 0)} \Delta z = 0$$

$$\Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} (f(x, y) - f(x_0, y_0)) = 0$$

$$\Rightarrow \lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0) \Leftrightarrow (2)$$