Partial derivatives

Def: $$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$

Notation: $$\frac{\partial f}{\partial x} = f_x = \partial_x f = \partial_1 f$$

$$\frac{\partial f}{\partial y} = f_y = \partial_y f = \partial_2 f$$

Eg 1. $f(x, y) = x^2 + 3xy + y - 1$

$$\partial_x f = 2x + 3y + 0$$

$$\partial_y f = 0 + 3x + 1$$

Eg2 $f(x,y,z) = x\sin(y+3z)$

$$\partial_x f \underset{y,z \sim \text{constant}}{=\!=} \sin(y+3z)$$

$$\partial_y f \underset{x,z \sim \text{const}}{=\!=} x\cos(y+3z)$$

$$\partial_z f \underset{x,y \sim \text{const}}{=\!=} x\underbrace{\cos(y+3z)}_{\sin'} \cdot \underset{\partial_z(y+3z)}{\underset{\parallel}{3}}$$

Eg3 Find $\frac{\partial z}{\partial x}$ at $(0,1,1)$

if $z(x,y)$ is implicitly defined by $yz+\ln z = x+y$

<u>Ans</u>: $y \cdot z(x,y) + \ln z(x,y) = x+y$

$$\partial_x \Rightarrow y \cdot z_x + \frac{z_x}{z} = 1+0$$

$$\Rightarrow \frac{\partial z}{\partial x}(x,y) = \frac{1}{y+\frac{1}{z}}$$

Check $(0,1,1)$ is on this surface

(Yes: $1 \cdot 1 + \ln 1 = 0+1$)

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{(0,1,1)} = \frac{1}{1+\frac{1}{1}} = \frac{1}{2}$$

Higher order Partial derivatives

$z = f(x,y)$

$$\frac{\partial^2 f}{\partial x^2} = \partial_x^2 f = \partial_x(\partial_x f) = f_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} = \partial_y^2 f = \partial_y(\partial_y f) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \partial_x(\partial_y f) = f_{yx}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \partial_y(\partial_x f) = f_{xy}$$

Similarly $\partial_y \partial_x^2 f = f_{xxy}$, etc.

Eg4: $f(x,y) = x^2 + y^2$

$\partial_x f = 2x$, $\partial_y f = 2y$

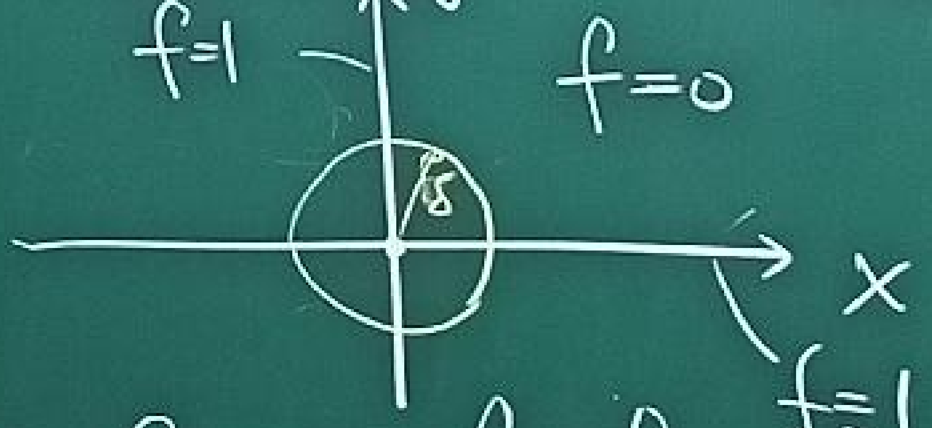
$\partial_x^2 f = \partial_x(2x) = 2$, $\partial_y^2 f = 2$

$\partial_x \partial_y f = \partial_x(2y) = 0$, $\partial_y(\partial_x f) = \partial_y(2x) = 0$

Eg 5. $\partial_x f(x_0, y_0)$, $\partial_y f(x_0, y_0)$ exist $\not\Rightarrow$ f is cont. at (x_0, y_0)

$$f(x,y) = \begin{cases} 0 & xy \neq 0 \\ 1 & xy = 0 \end{cases}$$

(i) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.



(ii) $\partial_x f(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x-0}$

$= \lim_{x\to 0} \frac{1-1}{x-0} = 0$

(iii) $\partial_y f(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y-0} = 0$

<u>Def</u> $z = f(x,y)$ is differentiable at (x_0, y_0) if

(i) $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist.

(ii) $f(x,y) = L(x,y) + \varepsilon_1 \cdot (x-x_0) + \varepsilon_2 \cdot (y-y_0)$

$\left(\text{or} = L(x,y) + \underline{\varepsilon} \cdot \sqrt{(x-x_0)^2 + (y-y_0)^2}\right)$

where

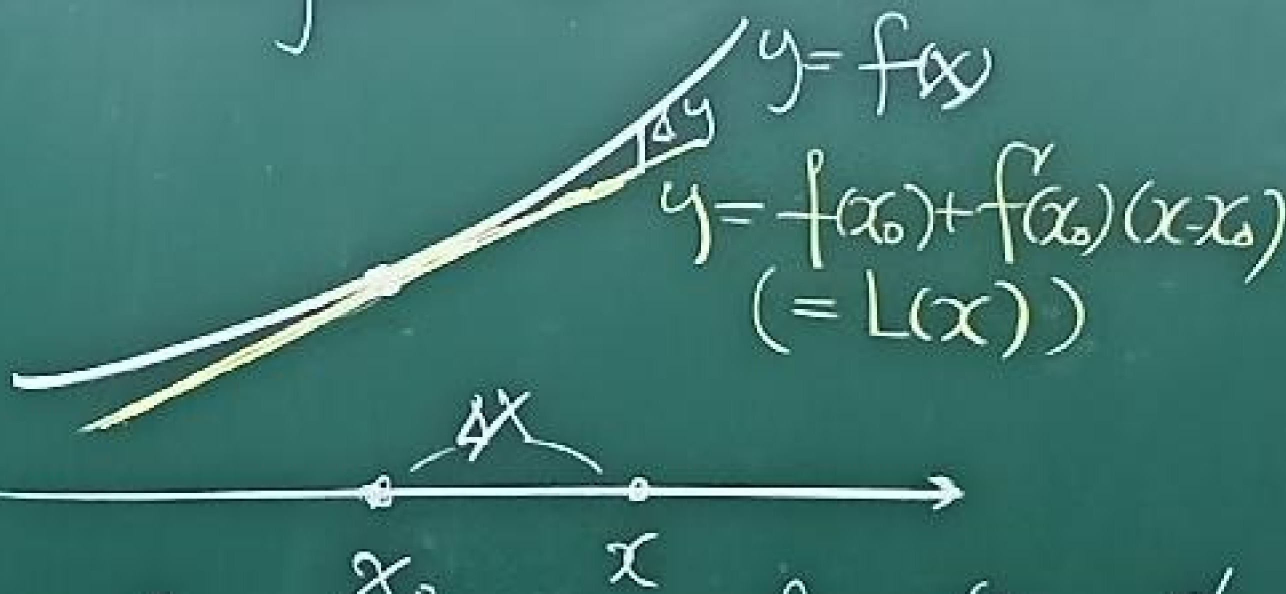
$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

where $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \varepsilon_1, \varepsilon_2, \underline{\varepsilon} = 0$

i.e. $f(x,y)$ and $L(x,y)$ are tangent at $(x_0, y_0, f(x_0,y_0))$

i.e. $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0 \left(= \lim_{(x,y)\to(x_0,y_0)} \underline{\varepsilon}\right)$$

Remark differentiability and tangent line in 1D



$$\lim_{x\to x_0}\frac{f(x)-L(x)}{x-x_0}=\lim_{x\to x_0}\frac{f(x)-(f(x_0)+f'(x_0)(x-x_0))}{x-x_0}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\qquad\qquad \parallel\ 0$$

Rm Two curves $\begin{cases} y=f(x) \\ y=g(x)\end{cases}$ are tangent at (x_0,y_0)

if (i) $f(x_0)=g(x_0)=y_0$

(ii) $\displaystyle\lim_{x\to x_0}\frac{f(x)-g(x)}{x-x_0}=0$ $\left(\begin{array}{c}|f(x)-g(x)| \text{ is} \\ \text{smaller than } |x-x_0|\end{array}\right)$

Rm (homework)

$$\varepsilon_1(x-x_0)+\varepsilon_2(y-y_0)=\varepsilon\cdot\sqrt{(x-x_0)^2+(y-y_0)^2}$$

Eg6. Are $\begin{cases} y=f_1(x)=e^x \\ y=f_2(x)=1+x \end{cases}$ tangent at $(0,1)$?

Sol: (i) $f_1(0)=1,\ f_2(0)=1$

(ii) $$\lim_{x\to 0}\frac{f_1(x)-f_2(x)}{x-0}=\lim_{x\to 0}\frac{e^x-1-x}{x}=0 \text{ (Yes)}$$

Eg7: Are $\begin{cases} z=f_1(x,y)=x^2+y^2 \\ z=f_2(x,y)=0 \end{cases}$ tangent at $(0,0,0)$?

Sol: (i) $f_1(0,0)=0,\ f_2(0,0)=0$

(ii) $$\lim_{(x,y)\to(0,0)}\frac{f_1(x,y)-f_2(x,y)}{\sqrt{x^2+y^2}}=\lim_{(x,y)\to(0,0)}\frac{x^2+y^2}{\sqrt{x^2+y^2}}=0 \text{ (Yes)}$$