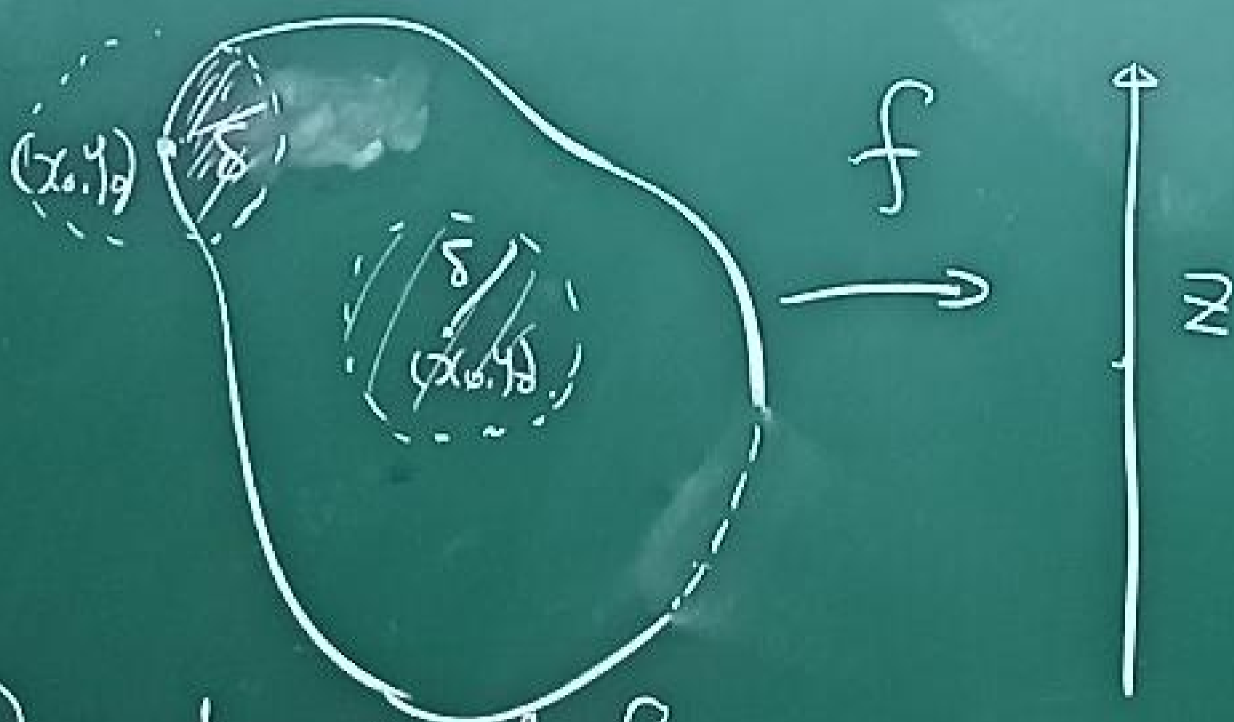
Limit and Continuity in higher dimension

2D case



D_f = domain of f

$$f : D_f \longmapsto \mathbb{R}$$

$$(x, y) \longmapsto z = f(x, y)$$

<u>Def</u> $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L$

if for every $\varepsilon > 0$

there exists a corresponding

$\delta > 0$, such that

" $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon$ "

$+ \left((x,y) \in D_f\right)$ $(*)$

<u>Remark</u>: If $(*)$ is changed to

" $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow |f(x,y) - L| < \varepsilon$ " $(**)$

then $\begin{cases} \lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = L \\ f(x_0,y_0) = L \end{cases}$

<u>Def</u> f is cont. at (x_0, y_0)

if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

That is, for every $\varepsilon > 0$,

there exists a corresponding

$\delta > 0$, such that

" $0 < \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta \Rightarrow |f(x,y) - f(x_0,y_0)| < \varepsilon$ "

$\left(\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta\right)$

<u>Limit Rules</u>: See Section 14.2

Theorem 1

Eg 1 $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x + y^2} = ?$

$\underline{\text{Ans}} = \dfrac{0 - 0\cdot 1 + 3}{0 + 1^2} = 3$

Here we have used the facts that, x, xy, y^2 are continuous.

Eg 2 $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = ?$

$$= \lim \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \, \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}$$

$$= \lim x(\sqrt{x} + \sqrt{y}) = 0$$

$\underline{\text{Rm}}$. $D_f = \{x \geq 0,\ y \geq 0,\ x \neq y\}$

Eg 3. $\displaystyle\lim_{\substack{(x,y)\to(0,0) \\ \text{"}r\to 0\text{"}}} \frac{4xy^2}{x^2+y^2} = ?$

"$0<r<\delta \Rightarrow |f(x,y)-?|<\varepsilon$"

<u>Ans</u>: $x = r\cos\theta$, $y = r\sin\theta$.

$$= \lim_{r\to 0} \frac{4r^3\cos\theta\sin^2\theta}{r^2} = 0$$

<u>Rm</u>

"$0<\sqrt{x^2+y^2}<\delta \Longrightarrow |f(x,y)-L|<\varepsilon$"

$\Longleftrightarrow$

"$0<r<\delta \Longrightarrow |f(r\cos\theta, r\sin\theta)-L|<\varepsilon$"

i.e. $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0} f(r\cos\theta, r\sin\theta)$

Eg4 $\lim_{(x,y)\to(0,0)} \frac{y}{x} = ?$

Ans: $x = r\cos\theta$, $y = r\sin\theta$

$$\lim_{r\to 0} \frac{r\sin\theta}{r\cos\theta} \overset{?}{=} L \quad (\text{indep of } \theta)$$

independent of

$$= \lim_{r\to 0} \frac{\sin\theta}{\cos\theta} \text{ does not exist.}$$

or Two Path Theorem (See below)

$$\lim_{(x,y)\to(0,0),\, x=y} \frac{y}{x} = 1$$

$$\lim_{(x,y)\to(0,0),\, x=-y} \frac{y}{x} = -1$$

$\Longrightarrow \lim_{(x,y)\to(0,0)}$ does not exist

Two Path Theorem

If $f(x,y)$ have different limits (one dimensional limits) along two Paths passing through (x_0, y_0) as $(x,y) \to (x_0, y_0)$

Then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist

Eg5. $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{x^2+y^2}$

See Fig 14.14 page 819

Sol. Let $(x,y)\to(0,0)$

along $y=mx$, $m\in\mathbb{R}$ (*)

(paths of straight lines)

$$\lim_{(*)} \frac{2xy}{x^2+y^2} = \lim_{x\to 0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

different $m\in\mathbb{R}$ give different one-dimensional limit.

Two Path Thm $\Rightarrow$ $\lim_{(x,y)\to(0,0)}$ does not exist

Eg6. $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2} = ?$

See Fig 14.15 page 820

(paths of straight lines passing (0,0))

Sol. Let $y = mx,\ x \to 0$

$$\lim_{x\to 0} \frac{2x^2 mx}{x^4+m^2x^2} = \begin{cases} 0, & m=0 \text{ (line "}y=0\text{")} \\ 0, & m\neq 0 \\ 0, & m=\infty \text{ (i.e. line "}x=0\text{")} \end{cases}$$

However,

if we let $y = kx^2,\ x \to 0$

(paths of parabolas)

$$\lim_{x\to 0} \frac{2kx^4}{x^4(1+k^2)} = \frac{2k}{1+k^2} \quad \left(\begin{array}{l}\text{different } k \\ \text{different lim}\end{array}\right)$$

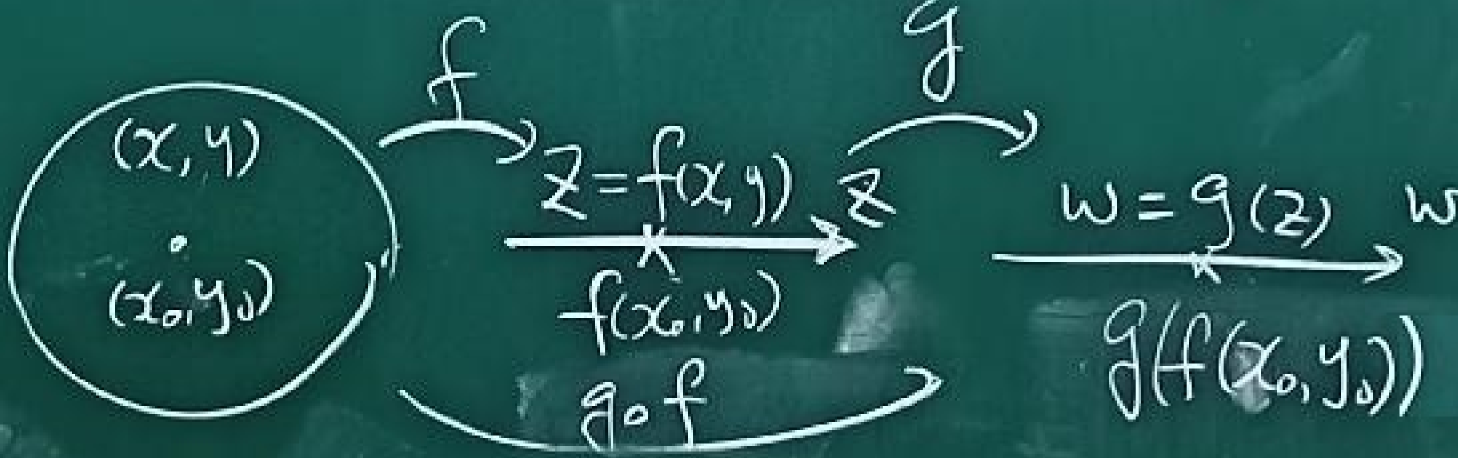
$\therefore$ Two Path Thm $\Rightarrow \lim_{(x,y)\to(0,0)}$ does not exist

Remark If

$$\begin{cases} f \text{ is cont. at } (x_0, y_0) \\ g \text{ is cont. at } z_0 = f(x_0, y_0) \end{cases}$$

Then $g \circ f$ is cont. at (x_0, y_0)

$$\left(g \circ f(x,y) \overset{\text{def}}{=\!=} g(f(x,y)) \right)$$



Eg7 $\cos\left(\frac{xy}{x^2+1}\right)$ is continuous on $\mathbb{R}^2$

$$(x,y) \xrightarrow{\ f\ } z = \frac{xy}{x^2+1} \xrightarrow{\ g\ } W = \cos z$$

Application:

$$\lim_{(x,y)\to(x_0,y_0)} \cos\left(\frac{xy}{x^2+1}\right) = \cos\left(\frac{x_0 y_0}{x_0^2+1}\right)$$