Eg5 (App II)

$$\lim_{x\to 0}\frac{1}{x}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$$

$$=\lim_{x\to 0}\frac{x-\sin x}{x^2\sin x}$$

$$=\lim_{x\to 0}\frac{x-\left(x-\frac{x^3}{3!}+\cdots\right)}{x^2\left(x-\frac{x^3}{3!}+\cdots\right)}$$

$$=\lim_{x\to 0}\frac{\frac{x^3}{3!}}{x^3-\frac{x^5}{3!}+\cdots}=\frac{1}{6}$$

Eg6 (App III, find $f(x)$ from $\sum a_n(x-a)^n$)

$$\frac{1}{1\cdot 2^1}-\frac{1}{2\,2^2}+\frac{1}{3\cdot 2^3}-\frac{1}{4\cdot 2^4}+\cdots=?$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\Big|_{x=\frac{1}{2}}$$

$$=-\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n}=-\sum_{n=1}^{\infty}\frac{(-x)^n}{n}\Big|_{x=\frac{1}{2}}$$

$$=\int_0^x\sum_{n=1}^{\infty}(-t)^{n-1}\,dt\Big|_{x=\frac{1}{2}}$$

$$=\int_0^x(1-t+t^2-t^3+\cdots)\,dt\Big|_{x=\frac{1}{2}}$$

$$=\int_0^{\frac{1}{2}}\frac{1}{1+t}\,dt=\ln(1+t)\Big|_0^{\frac{1}{2}}=\ln\left(\frac{3}{2}\right)$$

Method 2:

or Let $f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \quad (*)$

$\Rightarrow f'(x) = 1 - x + \frac{x^2}{\cancel{2}} - \frac{x^3}{\cancel{3}} + \cdots = \frac{1}{1+x}$

($|x| < 1$, ratio or root test)

$\Rightarrow f(x) = \ln(1+x) + C$

$(*)_{x=0} \Rightarrow f(0) = 0 = C$

$\therefore f(x) = \ln(1+x)$

We want $f(\frac{1}{2}) = \ln(\frac{3}{2})$

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Eg 7 $\sum_{n=1}^{\infty} n^2 x^n = ?$ $\left(\begin{array}{l}\text{See lecture 08} \\ |x|<1\end{array}\right)$

Eg 8 $\sum_{n=1}^{\infty} \frac{n(n+1)}{x^n} = ?$ $|x|>1$

$$= \sum_{n=1}^{\infty} n(n+1) y^n, \quad y = \frac{1}{x}, \quad |y|<1$$

$$f(y) = \sum_{n=1}^{\infty} y^n = \frac{1}{1-y} - 1 = \frac{-y}{1-y}$$

$$y f'(y) = \sum_{n=1}^{\infty} n y^{n-1} \cdot y$$

$$y^2 f''(y) = \sum_{n=1}^{\infty} n(n-1) y^{n-2} \cdot y^2$$

$$\text{Ans} = \sum_{n=1}^{\infty} n(n+1) y^n = 2y f'(y) + y^2 f''(y)$$

$$= \frac{2y}{(1-y)^2} + \frac{2y^2}{(1-y)^3} = \frac{2y}{(1-y)^3}$$

$$y = \frac{1}{x} \quad = \frac{2x}{(x-1)^3} \quad |x|>1$$

Remark (Euler identity)

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}$$

$i = \sqrt{-1}$

Since

$$e^{x} \overset{\text{def}}{=\!=} \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

can be identically defined for $z \in \mathbb{C}$ (complex number)

The radius of convergence $= R \left(\begin{matrix}\text{here}\\ =\infty\end{matrix}\right)$

means the series $\begin{cases} \text{converges on } |z|<R \\ \text{diverges on } |z|>R \end{cases}$

$\because e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$

Converges for all $z \in \mathbb{C}$

$$\therefore e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$\left(\overset{\text{check}}{\Longrightarrow} \frac{d}{d\theta} e^{i\theta} = ie^{i\theta} = i(\cos\theta + i\sin\theta) \right)$$

All algebraic computations and diff/integ. can be performed on $e^{i\theta}$ directly

Eg $\int e^{ax}\cos bx\,dx \quad a, b \in \mathbb{R}$

$$= \int e^{ax}\,\mathrm{Re}(e^{ibx})\,dx$$

$$= \mathrm{Re}\left(\int e^{ax} e^{ibx}\,dx\right)$$

$$= \mathrm{Re}\left(\int e^{(a+ib)x}\,dx\right)$$

$$= \mathrm{Re}\left(\frac{1}{a+ib}e^{(a+ib)x} + C\right)$$

$$= \mathrm{Re}\left(\frac{(a-ib)e^{(a+ib)x}}{(a+ib)(a-ib)} + C\right) = \frac{e^{ax}(a\cos bx + a\sin bx)}{a^2+b^2} + C$$