Binomial Series

$f(x) = (1+x)^m, \ m \in \mathbb{R}$

$T_{f,0}(x) = ? \stackrel{?}{=} f(x)$

(1) If $m \in \mathbb{N}$, $f(x)$ = polynomial

$\Rightarrow T_{f,0}(x) = f(x)$

(2) If $m \notin \mathbb{N}$

$f^{(k)}(x) = m(m-1)\cdots(m-k+1)(1+x)^{m-k}$

$f^{(k)}(0) = m(m-1)\cdots\cdots(m-k+1)$

$\Rightarrow f(x) = P_n(x) + R_n(x)$

where $P_n(x) = \sum_{k=0}^{n} \binom{m}{k} x^k$

$R_n(x) = \binom{m}{n+1} (1+C_{n+1})^{m-n-1} x^{n+1}$

$\therefore T_{f,0}(x) = \sum_{k=0}^{\infty} \binom{m}{k} x^k$

where $\binom{m}{k} = \dfrac{m(m-1)\cdots(m-k+1)}{k!}$

(i) Does $T_{f,0}(x)$ converge?

Ratio Test $\Rightarrow \rho = |x|$

(m fixed, $k \to \infty$)

$\therefore T_{f,0}(x)$ converges on $|x|<1$, diverges on $|x|>1$

(ii) $T_{f,0}(x) \stackrel{?}{=} f(x)$ on $|x|<1$?

$\lim R_n(x) = 0$?

Ans: Not clear if $-1<x<0$

It can be shown indirectly (and not easily) that

$$\lim_{n\to\infty} R_n(x) = 0 \quad \text{on } |x|<1$$

(Section 10.10, problem 58)

$$\therefore \underline{(1+x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k}$$

for all $m \in \mathbb{R}$, $|x|<1$

Important! (有!)

Known Taylor Series (암기!)

* $\dfrac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \cdots, \quad |x| < 1$
* $e^{\pm x} = 1 \pm x + \dfrac{x^2}{2!} \pm \dfrac{x^3}{3!} + \cdots \quad x \in \mathbb{R}$
* $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots \quad x \in \mathbb{R}$
* $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots \quad x \in \mathbb{R}$
* $\ln(1 \pm x) = \pm x - \dfrac{x^2}{2} \pm \quad -1 < x \leq 1$
* $\tan^{-1} x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots \quad |x| \leq 1$
* $(1 \pm x)^m = 1 \pm mx + \dfrac{m(m-1)}{2!} x^2 \pm \dfrac{m(m-1)(m-2)}{3!} x^3 + \cdots \quad |x| < 1$
* $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0 & x = 0 \end{cases}, \quad T_{f,0}(x) = 0 \neq f(x) \quad x \neq 0$

Eg1 $T_{\sin^{-1},0}(x) = ?$

Sol $(\sin^{-1}x)' = \frac{1}{\sqrt{1-x^2}}, \quad |x|<1$

$$\sin^{-1}x - \sin^{-1}0 = \int_0^x (\sin^{-1}t)'\,dt$$

(Fundamental Thm of Calc.)

$$= \int_0^x (1-t^2)^{-\frac{1}{2}}\,dt \quad (\text{Binomial}, \ m = -\tfrac{1}{2})$$

$$= \int_0^x \left(1 - \frac{1}{2}(-t^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-t^2)^2 + \cdots\right)dt$$

$$= x + \frac{x^3}{6} + \frac{x^5}{40} + \cdots, \quad |x|<1$$

$$(\text{Thm A}) = T_{\sin^{-1},0}(x)$$

Eg2 $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots, |x|<1$

$$\frac{\pi}{4} = \tan^{-1} 1 \overset{?}{=} 1 - \frac{1}{3} + \frac{1}{5} - \cdots ?$$

<u>Sol</u> $\frac{1}{1+t^2} = 1 - t^2 + \cdots + (-1)^n t^{2n}$

(<u>not</u> Taylor's Thm) $+ \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}, \forall t \in \mathbb{R}$

$\left(\because 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1-x}, x = -t^2\right)$

$$\Rightarrow \tan^{-1} x - \tan^{-1} 0 = \int_0^x \frac{1}{1+t^2} dt$$

$$= \int_0^x \left(1 - t^2 + t^4 - \cdots + (-1)^n t^{2n} + \frac{(-1)^{n+1} t^{2n+2}}{1+t^2}\right) dt$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^n x^{2n+1}}{2n+1} + \widetilde{R}_n(x)$$

Here $\tilde{R}_n(x) = \int_0^x \frac{(-1)^{n+1} t^{2n+2}}{1+t^2} dt$

If $x = 1$.

$$|\tilde{R}_n(1)| = \int_0^1 \frac{t^{2n+2}}{1+t^2} dt$$

$$\leq \int_0^1 \frac{t^{2n+2}}{1+0} dt = \frac{1}{2n+3}$$

(Similarly for $x = -1$)

$$\Rightarrow \lim_{n\to\infty} \tilde{R}_n(\pm 1) = 0$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

Leibnitz formula for π

Similarly $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Remark: If $|t|<1$

$$\frac{1}{1+t^2}=1-t^2+t^4 \cdots\cdots$$

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If $|t|>1$

$$\frac{1}{1+t^2}=\frac{1}{t^2}\left(\frac{1}{1+\frac{1}{t^2}}\right)$$

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$$=\frac{1}{t^2}\left(1-\frac{1}{t^2}+\frac{1}{t^4}-\cdots\cdots\right);\ |t|>1$$

Applications:

(I) Approximation and error estimate

(II) Altenative method for "$\lim \frac{0}{0}$"

(*)(III) Find $f(x)$ from $T_{f,a}(x)$

Eg3 (Application I)

Find app. value of $\int_0^{\frac{1}{2}} \sin t^2 \, dt$ and estimate the error.

<u>Ans</u>: $\int_0^{\frac{1}{2}} \sin t^2 \, dt$

$$= \int_0^{\frac{1}{2}} \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} \cdots\right) dt$$

$$= \frac{t^3}{3} - \frac{t^7}{7 \cdot 3!} + \frac{t^{11}}{11 \cdot 5!} \cdots \Big|_0^{\frac{1}{2}}$$

(Altenating Series

$$= \underbrace{\frac{1}{3 \cdot 2^3} - \frac{1}{7 \cdot 3! \cdot 2^7} + \frac{1}{11 \cdot 5! \, 2^{11}}}_{\text{Approximation}} - \cdots$$

$$|E| \leq \frac{\left(\frac{1}{2}\right)^{15}}{15 \cdot 7!}$$ (error estimate of Alternating Series)

Eg 4 (App II)

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3} = ?$$

Sol. Method 1: l'Hôpital.

Method 2:

$$f(x) = \frac{\sin x - \frac{\sin x}{\cos x}}{x^3}$$

$$= \frac{\frac{1}{2}\sin 2x - \sin x}{x^3 \cos x}$$

$$= \frac{\frac{1}{2}\left(\cancel{2x} - \frac{(2x)^3}{3!} + \cdots\right) - \left(\cancel{x} - \frac{x^3}{3!} + \cdots\right)}{x^3\left(1 - \frac{x^2}{2!} + \cdots\right)}$$

$$\lim_{x\to 0} = \frac{\cancel{x^3}\left(\frac{-4}{3!} + \frac{1}{3!} + \cdots\right)}{\cancel{x^3}\left(1 + \cdots\right)}$$

$$\Rightarrow \text{Ans} = \frac{-1}{2}$$