Taylor Series

Question: For a given function $f(x)$ and $a \in \mathbb{R}$, can we always find $a_k \in \mathbb{R}$ and $R > 0$ such that

$$(*)\quad f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

on $|x-a| < R$?

Ans: Not Necessarily.
(only for some f, not all f)

Remark If $a_k \in \mathbb{R}$ and
$R > 0$ do exist, then
we must have $a_k = \frac{f^{(k)}(a)}{k!}$ (✡)
from term by term diff. Thm.
That is, (✡) is the only candidate
and it may or may not work!

Question

If $f^{(k)}(a)$ exist for all $k = 0, 1, 2, \dots$

Is it necessarily true that (*) holds with (✡) for some $R > 0$?

Ans: Not necessarily.

(Counter example below: i.e. $f(x) \neq \sum_{k=0} \frac{f^{(k)}(a)}{k!}(x-a)^k$ for $x \neq a$)

<u>Def</u>: The Taylor Series generated by f at $x=a$

$$T_{f,a}(x) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

(= McLaurin Series if $a=0$)

<u>Def</u>: The Taylor Polynomial of degree n generated by f at $x=a$: $P_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$

Eg 1: f is a polynomial.

$$f(x) = a_0 + a_1 x + \cdots + a_5 x^5$$

Find $P_{3,0}(x)$, $P_{5,0}(x)$, $P_{7,0}(x)$, $T_{f,0}(x)$

Ans: $f^{(k)}(0) = k!\, a_k \quad 0 \le k \le 5$

$f^{(l)}(0) = 0 \quad$ for $l > 5$

$$\Rightarrow P_{3,0}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$P_{5,0}(x) = P_{7,0}(x) = T_{f,0}(x) = f(x)$$

Rm $P_{3,0}(x) \neq P_{3,1}(x)$, $T_{f,0}(x) = T_{f,1}(x)$

$P_{5,0}(x) = P_{5,1}(x)$, $P_{7,0}(x) = P_{7,1}(x)$

In general, if $f(x)$ is a polynomial of degree n, then

$$P_{m,a}(x) = T_{f,a}(x) = f(x)$$

for all $m \geq n$.

$$\because f(x) = a_0 + a_1 x + \cdots + a_n x^n = P_{n,0}(x)$$
$$= b_0 + b_1(x-a) + \cdots + b_n(x-a)^n = P_{n,a}(x)$$
$$\left(b_k = \frac{f^{(k)}(a)}{k!}\right)$$

$$P_{m,a}(x) = P_{n,a}(x) \text{ if } m > n.$$

$$\therefore P_{m,a}(x) = f(x) = T_{f,a}(x).$$

Eg 2 $f(x) = e^x$. $T_{f,a}(x) = ?$

<u>Ans</u>: $f^{(k)}(a) = e^a$

$\therefore T_{f,a}(x) = e^a \sum_{k=0}^{\infty} \frac{(x-a)^k}{k!}$ ([illegible] !)

<u>Remark</u>: from ratio test

$\rho = 0 \Rightarrow R$ (for $\sum \frac{(x-a)^k}{k!}$) $= \infty$

$\therefore T_{f,a}(x)$ converges for any $x \in \mathbb{R}$

(In fact, $T_{f,a}(x) = e^x$ for any $x \in \mathbb{R}$ (later))

Eg 3: $T_{\cos(x),0}(x)$

Sol $\cos^{(n)}(0) = \,?$

$n=0$	$n=1$	$n=2$	$n=3$
$4, 8, \ldots$	$5, 9, \ldots$	$6, 10, \ldots$	$7, 11, \ldots$
$\cos 0$	$-\sin 0$	$-\cos 0$	$\sin 0$
$\parallel$	$\parallel$	$\parallel$	$\parallel$
1	0	-1	0

$$\Rightarrow T_{\cos(x),0}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$ ($2k+1$ 月!)

($= \cos x$ for all $x \in \mathbb{R}$ (later))

Similarly

$$T_{\sin(x),0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} \quad (背!)$$

Eg 4. $T_{\frac{1}{1-x},0}(x) = ?$

Sol: $f(x) = \frac{1}{1-x} = (1-x)^{-1}$

$f'(x) = +(1-x)^{-2}$, $f'(0) = 1$

$f''(x) = +2(1-x)^{-3}$ $f''(0) = 2!$

$f^{(k)}(x) = k!(1-x)^{-k-1}$ $f'''(0) = 3!$

$$\Rightarrow T_{\frac{1}{1-x},0}(x) = 1 + x + x^2 + x^3 + \cdots$$

<u>ThmA</u>: If $f(x)$ has a power series representation $\left(f(x)=\sum_{k=0}^{\infty} a_k (x-a)^k\right)$ on $|x-a|<R$, $R>0$

$$\underset{\substack{\text{Term by} \\ \text{Term diff}}}{\Longrightarrow} f^{(k)}(a) = k!\, a_k \quad k \in \mathbb{N}$$

(derivations)

$$\Rightarrow T_{f,a}(x) = \sum_{k=0}^{\infty} \frac{k!\, a_k}{k!}(x-a)^k = f(x)$$

on $|x-a|<R$

conclusion:

$$\Rightarrow f(x) = T_{f,a}(x) \text{ on } |x-a|<R$$

<u>Method 2 for Eg4:</u>

$$\because \frac{1}{1-x} = 1+x+x^2+\cdots \quad (R=1>0) \Rightarrow T_{\frac{1}{1-x},0}(x) = 1+x+x^2+\cdots$$

In the following example

$f^{(k)}(0) = 0$ for all k

$\Rightarrow T_{f,0}(x) \equiv 0$ but $f(x) \neq 0$
$(x \neq 0)$

Eg5 $f(x) = \begin{cases} e^{\frac{-1}{x^2}}, & x \neq 0 \\ 0, & x = 0, \end{cases}$ $T_{f,0}(x) = ?$

Ans: $f(0) = 0$

$f'(0) = \lim_{h \to 0} \frac{e^{\frac{-1}{h^2}} - 0}{h - 0}$ $\left(\begin{array}{l} \text{Not always} \\ = \lim_{h \to 0} f'(h) \end{array} \right)$

$\left(= \text{"}\frac{0}{0}\text{"}\right) = \lim_{h \to 0} \frac{\left(\frac{1}{h}\right)}{e^{\frac{1}{h^2}}} \left(\pm \text{"}\frac{\infty}{\infty}\text{"}\right)$

l'Hôpital $\lim_{h \to 0} \frac{-h^{-2}}{-2h^{-3} e^{\frac{1}{h^2}}} = \lim_{h \to 0} \frac{h}{2e^{\frac{1}{h^2}}} = 0$

$(**)$

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h - 0}$$

$$\overset{(**)}{=} \lim_{h \to 0} \frac{2h^{-3} e^{-\frac{1}{h^2}}}{h - 0}$$

$$= \lim_{h \to 0} \frac{2e^{-\frac{1}{h^2}}}{h^4}$$

$$= \lim_{h \to 0} \frac{2h^{-4}}{e^{\frac{1}{h^2}}}$$

$$\underset{\text{l'Hôpital}}{=} \text{(homework)}$$

$$= 0$$

In fact, $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$