If $A(x)=\sum_{n=0}^{\infty} a_n x^n$

$B(x)=\sum_{n=0}^{\infty} b_n x^n$

both converge (abs.) on $|x|<R$

Can we find Power series Representation of $\frac{A(x)}{B(x)}$ if $b_0 \neq 0$?

Sol. If $C(x)=\frac{A(x)}{B(x)}=\sum_{n=0}^{\infty} C_n x^n$

$\Rightarrow A(x)=B(x)\cdot C(x)$

$\Rightarrow a_0=b_0 C_0 \Rightarrow C_0=\frac{a_0}{b_0}$

$a_1=b_0 C_1+b_1 C_0 \Rightarrow C_1=\frac{1}{b_0}(\cdots)$

$a_2=b_0 C_2+b_1 C_1+b_2 C_0 \Rightarrow C_2=\frac{1}{b_0}(\cdots)$

$\cdots\cdots$

Alternatively, we can find c_n more efficiently by long division.

Eg 1. $A(x) = 1$

$$B(x) = 1 + x + \frac{x^2}{2} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(In fact $B(x) = e^x$)

Find first few terms of $\frac{A(x)}{B(x)}$

$$
\begin{array}{r|l}
 & 1 \quad -1 \quad +\frac{1}{2} \quad -\frac{1}{6} + \cdots \\
\hline
1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots & 1 + 0 + 0 + 0 + \cdots \\
 & \underline{1 + 1 + \frac{1}{2} + \frac{1}{6} + \cdots} \\
 & \quad -1 - \frac{1}{2} - \frac{1}{6} + \cdots \\
 & \quad \underline{-1 - 1 - \frac{1}{2} + \cdots} \\
 & \qquad \frac{1}{2} + \frac{1}{3} + \cdots \\
 & \qquad \underline{\frac{1}{2} + \frac{1}{2}} \\
 & \qquad\quad -\frac{1}{6} + \cdots
\end{array}
$$

Ans:

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \cdots$$

$(= e^{-x})$

$\underline{\text{Rm}}$ If $A(x), B(x)$

both conv. (abs.) on $|x|<R$

and $b_0 = B(0) \neq 0$.

$\Rightarrow \exists\, \delta > 0$ such that

$\frac{A(x)}{B(x)}$ computed above

Converges on $|x| < \delta$.

Eg2: $A(x)=1$, $B(x)=1-x$
$1+x^2$

both converge on $|x|<\infty$

but $\frac{A(x)}{B(x)} = 1+x+x^2+x^3+\cdots \quad (\delta=1)$

$1-x^2+x^4-x^6+\cdots \quad (\delta=1)$

In fact, delta = "the smallest |z| (length of the complex number z) on the complex plane such that B(z)=0"

Term by term differentiation

Thm If $f(x)=\sum_{n=0}^{\infty} C_n (x-a)^n$

converges (abs.) on $|x-a|<R$

Then: (1): f', f'', f''', ...

all exist on $|x-a|<R$

(2). $f'(x)=\sum_{n=1}^{\infty} n C_n (x-a)^{n-1}$

$f''(x)=\sum_{n=2}^{\infty} n(n-1) C_n (x-a)^{n-2}$

.....

all converge on $|x-a|<R$

Eg 3: $f(x)=\sum_{n=0}^{\infty} x^n = 1+x+x^2+\cdots$

$= \frac{1}{1-x}$ on $|x|<1$

what is the power series representation of $\frac{1}{(1-x)^2}$?

Sol $\frac{1}{(1-x)^2} = f'(x)$

$= 1' + x' + (x^2)' + \cdots + (x^n)' + \cdots$

$= 1 + 2x + \cdots + nx^{n-1} + \cdots$

$= \sum_{n=1}^{\infty} nx^{n-1}$ converges on $|x|<1$

Rm $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}} = ?$ Ans. $= \left(\sum_{n=1}^{\infty} nx^{n-1}\right)_{x=\frac{1}{2}}$

$= \left(\frac{d}{dx}\sum_{n=0}^{\infty} x^n\right)_{x=\frac{1}{2}} = \frac{1}{(1-x)^2}\Big|_{x=\frac{1}{2}} = 4$

Eg 4. $\displaystyle\sum_{n=1}^{\infty} n^2 x^n = ?$

Sol $\displaystyle\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad -1 \quad (|x|<1)$

$\displaystyle\frac{d}{dx} \Rightarrow \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \quad -2$

$\displaystyle\frac{d}{dx} \Rightarrow \frac{2}{(1-x)^2} = \sum_{\substack{n=2 \\ (n=1)}}^{\infty} n(n-1) x^{n-2} \quad -3$

$$n^2 x^n = x^2\left(n(n-1)x^{n-2}\right) + x\left(n x^{n-1}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} n^2 x^n = \sum_{\substack{n=1 \\ (n=2)}}^{\infty} n(n-1)x^n + \sum_{n=1}^{\infty} n x^n$$

$$= x^2 \cdot \left(\frac{1}{1-x}\right)'' + x\left(\frac{1}{1-x}\right)'$$

$$= \frac{x+x^2}{(1-x)^3} \quad \text{valid on } |x|<1$$

<u>Remark</u> Term by term differentiation may not be valid for other series

Eg5: $f(x) = \sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$ is not a power series

Since $|a_n| \leq \frac{1}{n^2} \Rightarrow f(x)$ converges on $x \in \mathbb{R}$

But $\sum_{n=1}^{\infty} n a_n = \sum_{n=1}^{\infty} \frac{n!}{n^2} \cos(n!x)$

diverges for any $x \in \mathbb{R}$.

<u>Thm</u> (term by term integration)

If $f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$

converges abs. on $|x-a|<R$

Then $\sum_{n=0}^{\infty} \frac{C_n}{n+1}(x-a)^{n+1}$ also

converges on $|x-a|<R$

and $\int f(x)dx = \sum_{n=0}^{\infty} \frac{C_n}{n+1}(x-a)^{n+1} + C$

In fact

$$\int_a^x f(t)\,dt = \sum_{n=0}^{\infty} \int_a^x C_n (t-a)^n\,dt$$

$$= \sum_{n=0}^{\infty} \frac{C_n}{n+1}(x-a)^{n+1}$$

Eg6. Evaluate

$$F(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$$

Sol. Radius of convergence:

Ratio
Root test: $F(x)$ converge if $|x|^2 < 1$

$\Rightarrow R = 1$

On $|x| < 1$, $F'(x) = 1 - x^2 + x^4 - \cdots = \frac{1}{1+x^2}$

$$F(x) = \int_0^x F'(t)\,dt = \int_0^x \frac{1}{1+t^2}\,dt$$

$$= \tan^{-1} x$$

Note: $\tan^{-1} x$ is defined for all $x \in \mathbb{R}$

but $\neq \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}$ if $|x| \geq 1$