We now show that the region of convergence of a Power Series must be an interval centered at the "center of the Power Series". The radius of the interval is called the radius of convergence, usually can be found by Ratio Test or Root Test.

Thm If $\sum_{n=0}^{\infty} a_n x^n$ $\left(\sum a_n (x-a)^n\right)$ converges at $x = c \neq 0$ $(x = c \neq a)$

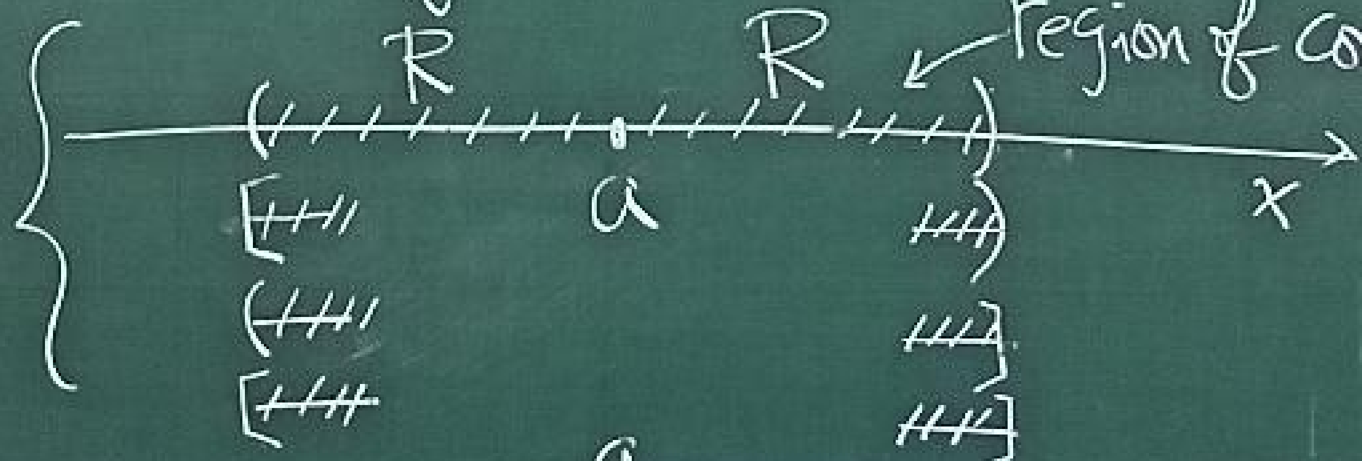
then it converges absolutely

for $|x| < |c|$ $\left(|x-a| < |c-a|\right)$

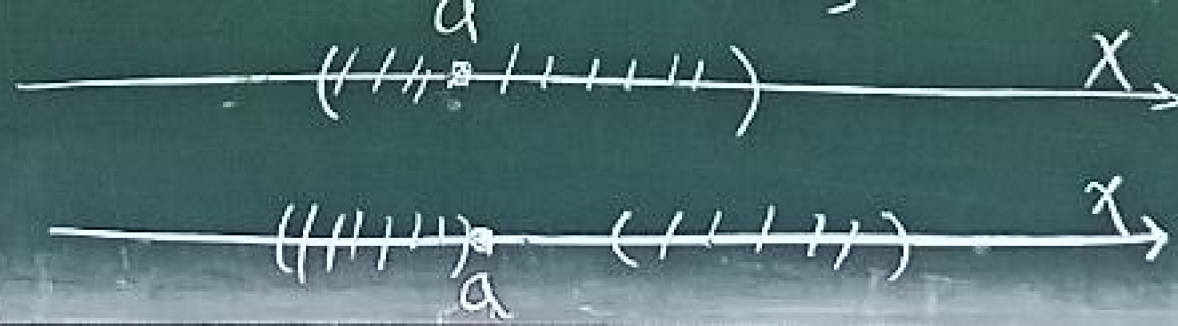
$\left(\Rightarrow\right.$ If it diverges at $x = d$,

then it diverges for $|x| > |d|$ $\left.\left(|x-a| > |d-a|\right)\right)$

Possible



Impossible



pf: If $\sum_{n=0}^{\infty} a_n c^n$ converges

$\Rightarrow \lim_{n\to\infty} a_n c^n = 0$

$\Rightarrow |a_n c^n| < 1 \quad \forall n \geq N$

$\Rightarrow |a_n| < \frac{1}{|c|^n} \quad \forall n \geq N$

If $|x| < |c|$

$\Rightarrow \sum_{n=N}^{\infty} |a_n x^n| \leq \sum_{n=N}^{\infty} \left(\frac{|x|}{|c|}\right)^n < \infty$

$\therefore \sum_{n=0}^{\infty} a_n x^n$ converges absolutely on $|x| < |c|$

In Summary: Possible regions of convergence.

(1) It converges absolutely for all $x \in \mathbb{R}$.

i.e. Radius of conv. $R = \infty$

(2) Convergence only at $x = a$ $(R = 0)$

(3) $\exists\ 0 < R < \infty$

$$\text{it} \begin{cases} \text{conv. on } |x| < R \\ \text{div. on } |x| > R \end{cases}$$

R = radius of conv. for $\sum a_n (x-a)^n$

In general $0 \le R \le \infty$

How to find R for $\sum a_n(x-a)^n$?

<u>Ans</u>: If $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = \rho$, $0 \le \rho \le \infty$.

$$\Rightarrow \lim_{n\to\infty} \frac{|a_n(x-a)^{n+1}|}{|a_n(x-a)^n|} = \rho\,|x-a|$$

$$\Rightarrow \sum a_n(x-a)^n \begin{cases} \text{conv. abs.} & \text{if } |x-a| < \frac{1}{\rho} \\ \text{div.} & \text{if } |x-a| > \frac{1}{\rho} \end{cases}$$

i.e. $R = \frac{1}{\rho}$

Similarly, if $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho$

Then $R = \frac{1}{\rho}$

Remark: R (radius of conv.) always exists for $\sum_{n=0}^{\infty} a_n (x-a)^n$

but $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ or $\lim_{n\to\infty} |a_n|^{\frac{1}{n}}$

may or may not exist.

Eg. $\sum_{n=1}^{\infty} a_n x^n$

$= \left(\frac{x}{2}\right)^1 + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{4}\right)^4 + \cdots$

$\lim \frac{|a_{n+1}|}{|a_n|}$, $\lim |a_n|^{\frac{1}{n}}$ do not exist

Rm: $R = \left(\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}\right)^{-1} = 2$

Eg 1 ⓐ $\sum_{n=0}^{\infty} x^n$ conv. on $(-1,1)$
div. elsewhere

ⓑ $\sum_{n=0}^{\infty} \frac{x^n}{n}$ conv. on $[-1,1)$
div. elsewhere

$a_n = \frac{1}{n} \xrightarrow[\text{Root}]{\text{Ratio}} \rho = 1 \Rightarrow R = 1$

$|x|<1$. conv. $|x|>1$. div

$x=1$. p-series, $p=1$

$x=-1$ Alternating Series test

ⓒ $\sum_{n=0}^{\infty} \frac{(-x)^n}{n}$ conv on $(-1,1]$
div elsewhere.

ⓓ $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ conv on $[-1,1]$
div elsewhere.

Algebraic Manipulation of Power Series.

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$B(x) = \sum_{n=0}^{\infty} b_n x^n$

both conv on $|x| < R$.

Then $A(x) \pm B(x) = ?$

$A(x) \cdot B(x) = ?$

$A(x) / B(x) = ?$

Thm If $A(x) = \sum_{n=0}^{\infty} a_n x^n$

$$B(x) = \sum_{n=0}^{\infty} b_n x^n$$

both conv (abs.) on $|x| < R$

and $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$

$$\left(= \sum_{k=0}^{n} a_k b_{n-k}\right)$$

Then $\sum_{n=0}^{\infty} c_n x^n$ conv. abs. and $= A(x)B(x)$

on $|x| < R$.

Sol. $A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$

$$B(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$$

$$\Rightarrow A(x) \cdot B(x) = a_0 b_0 + (a_1 b_0) x + (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 + \cdots$$