<u>proof</u> (of The Ratio Test
The Root test is similar)

(1) $0 \leq \rho < 1$, $\left(\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \right)$

Take $r = \frac{1+\rho}{2} < 1$

Since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho$

Take $\varepsilon = r - \rho > 0$, there exists a corresponding $N \in \mathbb{N}$ such that

"$n > N \Rightarrow \left| \left| \frac{a_{n+1}}{a_n} \right| - \rho \right| < \varepsilon$"

$\left(\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| < \underbrace{\rho + \varepsilon}_{= r} \right)$

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{N}|a_n| + \sum_{n=N+1}^{\infty}|a_n|$$

$$= \sum_{n=1}^{N}|a_n| + |a_{N+1}| + |a_{N+2}| + \cdots$$

$$\leq (\;\text{''}\;) + |a_N| r + |a_N| r^2 + \cdots\cdots$$

$$= \left(\sum_{n=1}^{N}|a_n|\right) + \text{convergent Geometric Series} < \infty$$

(2) $\rho > 1$, define $r = \frac{1+\rho}{2} > 1$

let $\varepsilon = r - \rho > 0 \Rightarrow \exists N \in \mathbb{N}$

such that

"$n > N \Rightarrow \left|\left|\frac{a_{n+1}}{a_n}\right| - \rho\right| < \varepsilon$"

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| > \rho - \varepsilon = r > 1$$

$$\Rightarrow \lim |a_n| \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

Alternating Series test
(Leibnitz test)

If (1) $u_n > 0$

(2) $u_n \geqq u_{n+1}$

(for all $n \geqq N$)

(3) $\lim_{n \to \infty} u_n = 0$

Then $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges

Eg1: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ converges

Sol $u_n = \frac{1}{n} > 0$.

$u_n > u_{n+1}$, $\lim u_n = 0$

$\therefore$ Leibnitz test $\Rightarrow$ converges

Def (1) $\sum a_n$ converges <u>absolutely</u>

if $\sum |a_n| < \infty$

(2) $\sum a_n$ converges <u>conditionally</u>

if $\begin{cases} \sum a_n \text{ converges} \\ \sum |a_n| = \infty \end{cases}$

Eg2 ⓐ $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely.

ⓑ $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ $\quad p > 0$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{conv.} & p > 1 \\ \text{div} & 0 < p < 1 \end{cases}$

$\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$ converges $\begin{matrix} \text{abs.} & p > 1 \\ \text{cond.} & 0 < p < 1 \end{matrix}$

Eg 3 $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n\sqrt{\ln n}}$ conv abs? cond.?

Sol: $\sum (-1)^{n+1} u_n$ converges (Leibnitz Test)

Does $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converge?

Integral test:

$$\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx = \int_{x=2}^{\infty} \frac{1}{\sqrt{\ln x}}\,d\ln x$$

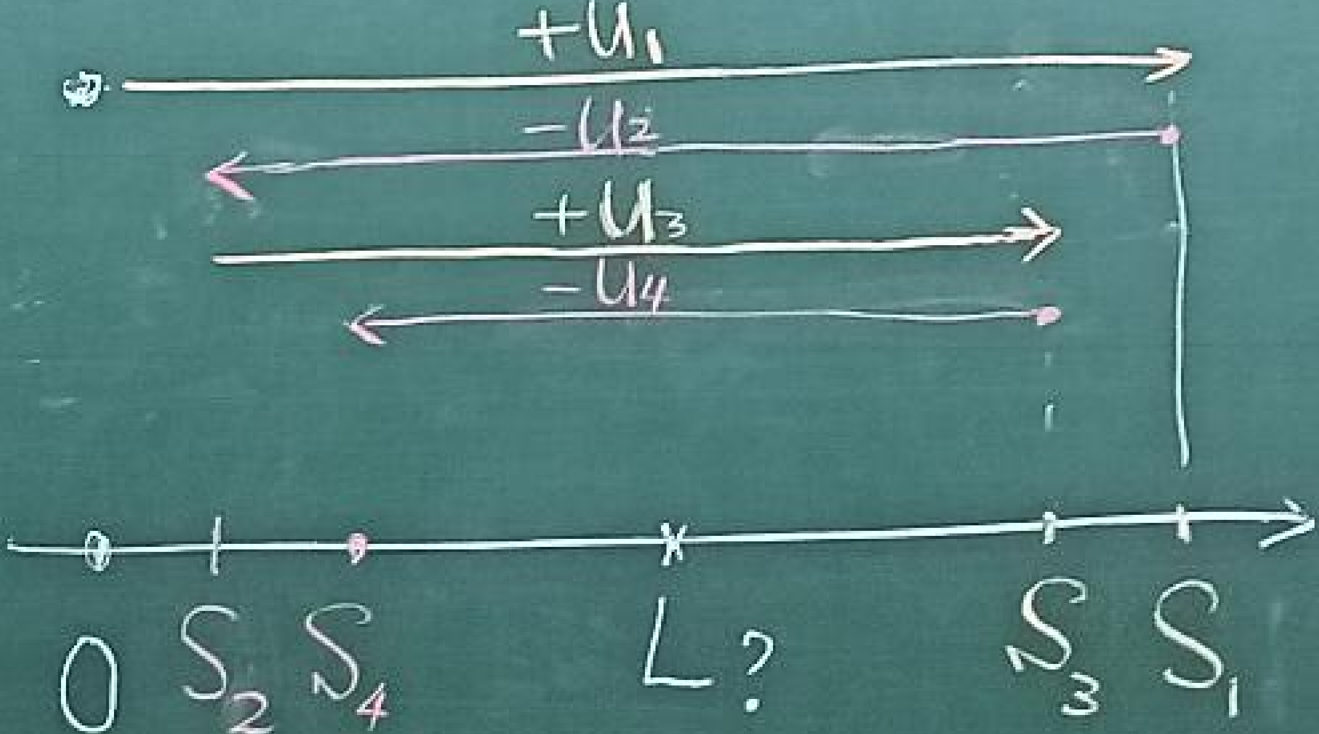
$$= \int_{y=\ln 2}^{\infty} \frac{1}{\sqrt{y}}\,dy \quad (y = \ln x)$$

$= $ "$p = \frac{1}{2}$" $=$ divergent.

Ans: $\sum \frac{(-1)^{n+1}}{n\sqrt{\ln n}}$ converges cond.

Pf of Leibnitz test

Let $S_n = \sum_{k=1}^{n} (-1)^{k+1} u_k$



$\Rightarrow 0 < S_2 < S_4 < \cdots\cdots < S_3 < S_1$

$\therefore \{S_2, S_4, \cdots \cdots S_{2k}, \cdots\cdots\}$

Is an increasing sequence.

and bounded above $(S_{2k} < S_1)$

From the Monotone Sequence Thm (Section 10.1. Thm 6)

$\lim_{k\to\infty} S_{2k} = L$ for some $L \in \mathbb{R}$

Moreover $S_{2k+1} = S_{2k} + u_{2k+1} \xrightarrow[k\to\infty]{} L + 0$

$\therefore \lim_{n\to\infty} S_n = L$

Remark: (1) We assumed for simplicity

$u_1 \geq u_2 \geq u_3 \geq \cdots$ (i.e. decreasing for all n)

(2) $S_{2k} < L < S_{2l-1}$ $\forall k, l \in \mathbb{N}$

$\Rightarrow 0 < L - S_{2k} < u_{2k+1}$

$0 < S_{2l-1} - L < u_{2l}$ (error estimate of partial sum)

Def. Power Series

$$\sum_{n=0}^{\infty} C_n(x-a)^n \overset{def}{=\!=} C_0 + \sum_{n=1}^{\infty} C_n(x-a)^n$$

It is a fantion of x

a = center (fixed). C_n = Coefficients

Question: When "a" and "C_n" are given, for what values of x is the power series convergent?

Eg 1. $\displaystyle\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n (x-2)^n$. (Geometric Series)

$r = \dfrac{-(x-2)}{2}$ (div if $|r| \geq 1$)

Power Series converges (abs) $\Leftrightarrow |r| < 1 \Leftrightarrow 0 < x < 4$

Eg2 $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$(0! \overset{\text{def}}{=} 1.\ x^0 \overset{\text{def}}{=} 1)$

Sol: Ratio test:

$|u_n| = \frac{|x|^n}{n!}$

$\lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x|}{n} = 0$

$\Rightarrow$ For any $x \in \mathbb{R}$

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges (absolutely)

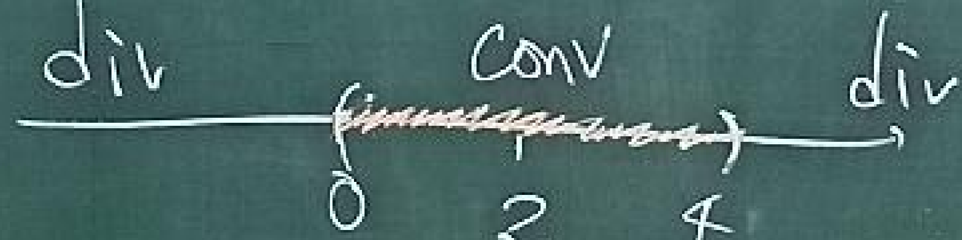
Eg3 $\sum_{n=0}^{\infty} n!\,x^n$

Ratio Test:

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} n|x| = \begin{cases} 0. & x=0 \\ \infty. & x\neq 0 \end{cases}$$

It converges at $x=0$ only and diverges elsewhere.

Eg1



Eg2

Eg3