Eg8 $\sum a_n$ $\sum b_n$ Thm Result

(g) $\displaystyle\sum_{n=1}^{\infty} \tan\left(\frac{1}{n^2}\right)$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\|$(i) $(C=1)$ conv

$\because \displaystyle\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)\left(\frac{1}{\cos\theta}\right) = 1$

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ $\|$(i) $(C=1)$ div.

$\because \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n^{\frac{1}{n}} = 1.$

$\sum a_n$ $\quad \sum b_n$ $\quad$ Thm $\quad$ Res.

(i) $\sum_{n=1}^{\infty} \sqrt{\frac{\ln n}{n}}$ $\quad \sum \frac{1}{\sqrt{n}}$ $\quad$ 10 $\quad$ div.

(j) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n \ln n}}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^p}$ $\quad \underset{(c=+\infty)}{11\,\text{(iii)}}$ div.

$p = \frac{1}{2} \longrightarrow$ inconclusive.

$p = 1, \frac{3}{4}, 0.9, 0.51, \cdots$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n \ln n}}}{\frac{1}{n}}$$

(take $p=1$) $\displaystyle = \lim_{n\to\infty} \left(\frac{n}{\ln n}\right)^{\frac{1}{2}} = \infty$

Since $\sum b_n = \sum \frac{1}{n} = \infty$

$\Rightarrow \sum a_n = \infty$

Ratio Test and Root Test

Thm (Ratio Test)

If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \rho$,

ⓐ $0 \le \rho < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

ⓑ $\rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ div.

ⓒ $\rho = 1 \Rightarrow$ inconclusive.

Thm $\sum_{n=1}^{\infty} |a_n| < \infty \Rightarrow \sum_{n=1}^{\infty} a_n$ conv.

pf: $-|a_n| \le a_n \le |a_n|$, $0 \le a_n + |a_n| \le 2|a_n|$

$\sum |a_n| < \infty \Rightarrow \sum 2|a_n| < \infty \Rightarrow \sum a_n + |a_n| < \infty$

$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n| =$ conv $-$ conv $=$ conv.

Thm (Root Test)

If $\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = \rho$

ⓐ $0 \le \rho < 1 \Rightarrow \sum_{n=1}^{\infty} |a_n| < \infty$

ⓑ $\rho > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ div.

ⓒ $\rho = 1 \Rightarrow$ inconclusive.

Eg 1
$$\begin{cases} \sum \frac{1}{n} = \infty & (\rho = 1) \\ \sum \frac{1}{n^2} < \infty & (\rho = 1) \end{cases}$$

case ⓒ for both ratio test and root test.

Eg2 $a_n = (-1)^{n+1}\frac{1}{n}$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$$\sum_{n=1}^{\infty} a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2k-1} - \frac{1}{2k} + \cdots$$

$$= \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} - \frac{1}{2k} \right)$$

$$= \sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} \quad \text{conv}$$

$\left(\text{Compare with } \sum_{k=1}^{\infty} \frac{1}{k^2} \right)$

Rm: This example shows:

$\left(\sum |a_n| = \infty \not\Rightarrow \sum a_n \text{ div} \right)$

Eg 3

(a) $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

Sol: Ratio Test

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim \frac{2^{n+1}+5}{3\cdot(2^n+5)}$$

$$= \lim_{n\to\infty} \frac{2+5\cdot 2^{-n}}{3\cdot(1+5\cdot 2^{-n})} = \frac{2}{3}$$

$\rho < 1 \Rightarrow$ convergent.

Root test:

$$\lim |a_n|^{\frac{1}{n}} = \lim \left(\frac{2^n(1+5\cdot 2^{-n})}{3^n}\right)^{\frac{1}{n}}$$

$$= \frac{2}{3} < 1$$

(b) $\sum_{n=1}^{\infty} \frac{(2n)!}{n!\,n!}$

Sol. Ratio Test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{(n+1)!\,(n+1)!}}{\frac{(2n)!}{n!\,n!}}$$

$$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} = 4 > 1$$

Ratio test $\Rightarrow$ divergent

ⓒ $\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$

<u>Sol</u>: Ratio Test

$\rho = 1$ (See ⓑ)

Ratio Test is inconclusive

However,

$$\frac{a_{n+1}}{a_n} \overset{ⓑ}{=} \frac{4(n+1)(n+1)}{(2n+1)(2n+2)}$$

$$= \frac{(2n+2)\cancel{(2n+2)}}{(2n+1)\cancel{(2n+2)}} > 1$$

$\Rightarrow a_{n+1} > a_n > 0$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0$, $\sum a_n$ div!

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n} \xrightarrow[\text{Root}]{\text{Ratio}} \rho = \frac{1}{2}$ conv.

(e) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right)^n \xrightarrow{\text{Root}} \rho = 0$ conv.

(f) $a_n = \begin{cases} \dfrac{n}{2^n} & n \text{ is odd} \\ \dfrac{1}{2^n} & n \text{ is even} \end{cases}$

$= \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{1}{16}, \frac{5}{32}, \frac{1}{64}, \frac{7}{128}, \frac{1}{256}, \cdots$

$\dfrac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{2} \cdot \frac{1}{n} & n \text{ is odd} \\ \frac{1}{2} \cdot (n+1) & n \text{ is even} \end{cases}$

$\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist : inconclusive.

However $\displaystyle\lim_{n\to\infty} a_n^{\frac{1}{n}} = \frac{1}{2} \Rightarrow \sum a_n$ converges.